Mean value theorems

Theorem Let $f:[a,b]\to\mathbb{R}$ be a continuous function.

(a) There exists a $c\in[a,b]$ such that if $x\in[a,b]$ then $f(x)\le f(c)$.

There exists a $d\in[a,b]$ such that if $x\in[a,b]$ then $f(d)\le f(x)$.

(b) If $c\in(a,b)$ ~~such~~ that if $x\in[a,b]$ then $f(x)\le f(c)$ ~~then~~ and $f'(c)$ exists then $f'(c)=0$.

(c) If $f(a)=f(b)$ and f is differentiable on (a,b) then there exists $c\in(a,b)$ such that $f'(c)=0$.

(d) If f is differentiable on (a,b) then there exists $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Theorem (Fancy version) Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be continuous functions such that f and g are differentiable on (a,b). Then there exists $c\in(a,b)$ such that

$$(f(b)-f(a))g'(c)=(g(b)-g(a))f'(c).$$

Proof Let $h:[a,b] \to \mathbb{R}$ be given by

$$h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x).$$

Then $h(a) = (f(b)-f(a))g(a) - (g(b)-g(a))f(a)$

$$= f(b)g(a) - g(b)f(a)$$

and

$$h(b) = (f(b)-f(a))g(b) - (g(b)-g(a))f(b)$$

$$= g(a)f(b) - f(a)g(b).$$

So $h(a) = h(b)$. and h is differentiable on (a,b).

So there exists $c \in (a,b)$ such that

$$h'(c) = 0.$$

So

$$0 = (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c).$$

So

$$(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c).$$

L'Hopitals rule

Assume $f:(a,b)\to\mathbb{R}$ and $g:(a,b)\to\mathbb{R}$ are differentiable.

Assume $g'(x)\neq 0$ if $x\in(a,b)$.

Assume $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$.

Assume $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

<u>Proof</u> Uses fancy version of Mean Value Theorem.

Works only for functions $f:(a,b)\to\mathbb{R}$ and $g:(a,b)\to\mathbb{R}$ with $g'(x)\neq 0$ for all $x\in(a,b)$.

The proof is not very conceptual.

<u>Example</u> $\lim_{x\to 0}\frac{5x}{x}=\lim_{x\to 0} 5=5.$

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots-1}{x}$$

$$=\lim_{x\to 0}\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}{x}=\lim_{x\to 0} 1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots$$

$$=\lim_{x\to 0} 1+x\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right)$$

By the ratio test

$\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$ converges for all $x \in \mathbb{R}$ (or $\mathbb{C}$).

Let $\ell = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$.

To show: $\lim_{x \to 0} 1 + x\left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right) = 1$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $|x - 0| < \delta$ then $\left|1 + x\left(\frac{1}{2} + \frac{x}{3!} + \cdots\right) - 1\right| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Let $\delta = \min\left(\frac{\varepsilon}{|\ell|}, 1\right)$. To show: $\left|1 + x\left(\frac{1}{2} + \frac{x}{3!} + \cdots\right) - 1\right| < \varepsilon$.

Then

$$\left|1 + x\left(\frac{1}{2} + \frac{x}{3!} + \cdots\right) - 1\right| = \left|x\left(\frac{1}{2} + \frac{x}{3!} + \cdots\right)\right|$$

$\leq |x| \cdot |\ell|$, since $|x| < 1$

$< \delta |\ell|$, since $|x| < \delta$

$< \varepsilon$.

POINT: Don't use L'Hopital's rule, use Taylor series.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + \frac{df}{dx}\Big|_{x=a}(x-a) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(x-a)^2 + \cdots}{g(a) + \left(\frac{dg}{dx}\Big|_{x=a}\right)(x-a) + \frac{1}{2!}\left(\frac{d^2g}{dx^2}\Big|_{x=a}\right)(x-a)^2 + \cdots}$$

and if $f(a) = 0$ and $g(a) = 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\left(\frac{df}{dx}\Big|_{x=a}\right)(x-a) + \left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(x-a)^2 + \cdots}{\left(\frac{dg}{dx}\Big|_{x=a}\right)(x-a) + \left(\frac{d^2g}{dx^2}\Big|_{x=a}\right)(x-a)^2 + \cdots}$$

$$= \lim_{x \to a} \frac{\left(\frac{df}{dx}\Big|_{x=a}\right) + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\Big|_{x=a}\right)(x-a) + \cdots}{\left(\frac{dg}{dx}\Big|_{x=a}\right) + \frac{1}{2!}\left(\frac{d^2g}{dx^2}\Big|_{x=a}\right)(x-a) + \cdots}$$

POINT 2 Use and learn limits that come up all the time:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}.$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1.$$