620-295 Real Analysis with applications, lect. 25

5 October 2009.

Integrals

Write

$$\int f(x)\,dx = g(x) + c, \quad \text{if} \quad \frac{d\,g(x)}{dx} = f(x).$$

The fundamental theorem of calculus says

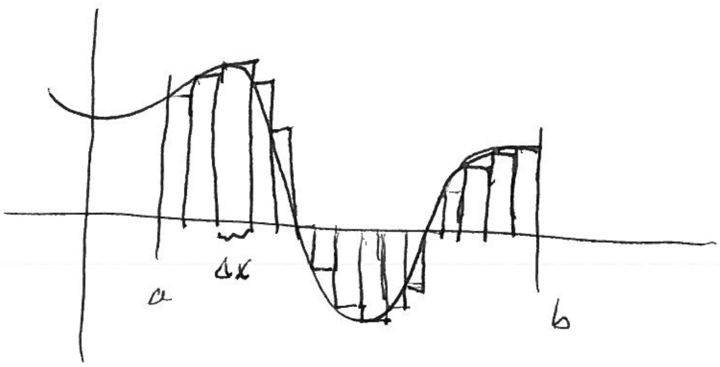
sometimes $\displaystyle\int_a^b f(x)\,dx = g(b) - g(a).$

BUT:

$\displaystyle\int_a^b f(x)\,dx$ really means

$$\lim_{\Delta x \to 0} \left(f(a)\Delta x + f(a+\Delta x)\Delta x + \cdots + f(b-\Delta x)\Delta x \right)$$

$$= \lim_{\Delta x \to 0} \left(\text{add up the areas of the little boxes } \underbrace{\square}_{\Delta x}\big\} f(a+\Delta x)\right)$$

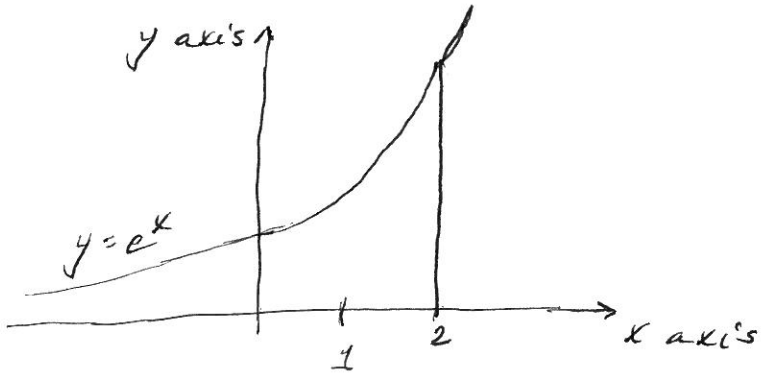


First box has area $f(a)\Delta x$,

Second box has area $f(a+\Delta x)\Delta x$, ...

②

Example $\int_0^2 e^x\,dx$



$$\int_0^2 e^x\,dx = \lim_{\Delta x \to 0} \left(e^0 \Delta x + e^{\Delta x}\Delta x + e^{2\Delta x}\Delta x + \cdots + e^{(2-\Delta x)}\Delta x \right)$$

If $\Delta x = \frac{1}{3}$, then

$$e^0\Delta x + e^{\Delta x}\Delta x + \cdots + e^{2-\Delta x}\Delta x$$

$$= e^0\tfrac{1}{3} + e^{\frac{1}{3}}\tfrac{1}{3} + e^{2/3}\tfrac{1}{3} + e^{3/3}\tfrac{1}{3} + e^{4/3}\tfrac{1}{3} + e^{5/3}\cdot\tfrac{1}{3}$$

$$= \tfrac{1}{3}\left(e^0 + e^{\frac{1}{3}} + (e^{\frac{1}{3}})^2 + (e^{\frac{1}{3}})^3 + (e^{\frac{1}{3}})^4 + (e^{\frac{1}{3}})^5\right)$$

$$= \tfrac{1}{3}\left(\frac{e^{6/3}-1}{e^{\frac{1}{3}}-1}\right) = (e^2-1)\left(\frac{\frac{1}{3}}{e^{\frac{1}{3}}-1}\right)$$

If $\Delta x = \frac{1}{5}$ then

$$e^0\Delta x + e^{\Delta x}\Delta x + \cdots + e^{2-\Delta x}\Delta x = e^0\tfrac{1}{5} + e^{\frac{1}{5}}\cdot\tfrac{1}{5} + \cdots + e^{9/5}\cdot\tfrac{1}{5}$$

$$= \tfrac{1}{5}\left(e^0 + e^{\frac{1}{5}} + (e^{\frac{1}{5}})^2 + (e^{\frac{1}{5}})^3 + \cdots + (e^{\frac{1}{5}})^9\right)$$

$$= \tfrac{1}{5}\left(\frac{e^{10/5}-1}{e^{\frac{1}{5}}-1}\right) = (e^2-1)\,\frac{\frac{1}{5}}{e^{\frac{1}{5}}-1}$$

So

$$\int_0^2 e^x\,dx = \lim_{\Delta x\to 0}\left(e^0\Delta x + \cdots + e^{2-\Delta x}\Delta x\right) = \lim_{N\to\infty}(e^2-1)\left(\frac{\frac{1}{N}}{e^{\frac{1}{N}}-1}\right)$$

Recall: $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

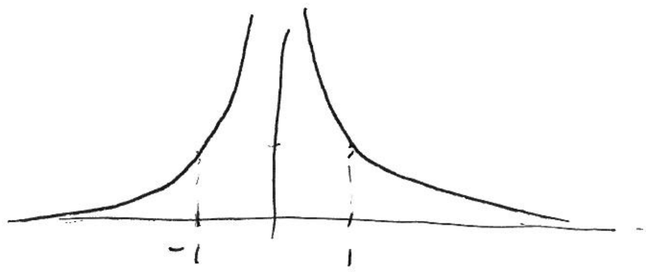
So $\displaystyle\lim_{N \to \infty} \frac{e^{1/N} - 1}{1/N} = 1.$ So $\displaystyle\lim_{N \to \infty} \frac{1/N}{e^{1/N} - 1} = 1.$

So
$$\int_0^2 e^x \, dx = \lim_{N \to \infty} (e^2 - 1)\left(\frac{1/N}{e^{1/N} - 1}\right) = (e^2 - 1) \cdot 1 = e^2 - 1.$$

Note: $\displaystyle\int e^x \, dx = e^x + c$ and

$$e^x + c \Big|_{x=0}^{x=2} = (e^2 + c) - (e^0 + c) = e^2 - 1.$$

<u>Example</u> $\int_{-1}^{1} \frac{1}{x^2}\,dx$



$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \lim_{\Delta x \to 0} \left(\frac{1}{(-1)^2}\Delta x + \frac{1}{(-1+\Delta x)^2}\Delta x + \cdots + \frac{1}{(1-\Delta x)^2}\Delta x \right)$$

$$= \lim_{\Delta x \to 0} \left(1\cdot\Delta x + \frac{1}{(-1+\Delta x)^2}\Delta x + \cdots + \frac{1}{0^2}\Delta x + \cdots + \frac{1}{(1-\Delta x)^2}\Delta x \right)$$

<u>oops</u>

So $\int_{-1}^{1} \frac{1}{x^2}\,dx$ is UNDEFINED. i.e. diverges.

<u>Note</u>: $\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx$ and

$$-x^{-1}+c \Big|_{x=-1}^{x=1} = (-1^{-1}+c) - (-(-1)^{-1}+c) = -1-1 = -2.$$

So this is a case where

$$\int_a^b \frac{df}{dx}\cancel{(x)}\,dx \neq f(b)-f(a)$$

i.e. adding up areas of little boxes

and

doing the indefinite integral and plugging in give <u>different</u> <u>answers</u>.

The fundamental theorem of calculus says

$$\begin{array}{c}\text{Area under } g(x)\\ \text{from } a \text{ to } b\end{array} = A(b) - A(a),$$

where $\int g(x)\,dx = A(x) + c$.

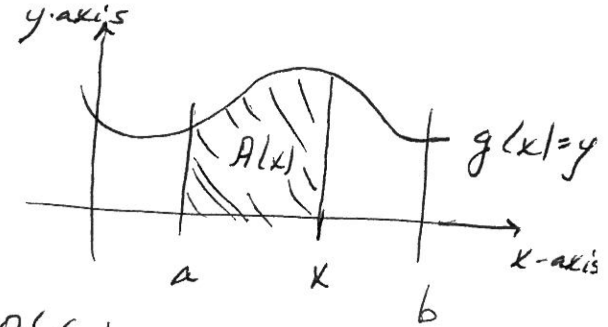
Idea of proof:

To show: There exists $A : [a,b] \to \mathbb{R}$ such that

(a) If $c \in [a,b]$ then $A'(c) = g(c)$,

(b) $A(b) - A(a) = \begin{array}{c}\text{Area under } g(x)\\ \text{from } a \text{ to } b.\end{array}$

Let $A(x) = \begin{array}{c}\text{area under } g(x)\\ \text{from } a \text{ to } x\end{array}$



To show: (a) If $c \in [a,b]$ then $A'(c) = g(c)$

(b) $A(b) - A(a) = $ Area under $g(x)$ from a to b.

(a) Assume $c \in [a,b]$.

To show: $A'(c) = g(c)$

$$A'(c) = \lim_{\Delta x \to 0} \frac{A(c+\Delta x) - A(c)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\begin{pmatrix}\text{area under } g(x)\\ \text{from } a \text{ to } c+\Delta x\end{pmatrix} - \begin{pmatrix}\text{area under } g(x)\\ \text{from } a \text{ to } c\end{pmatrix}}{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \frac{\text{area of last little box}}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(c)\,\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} g(c) = g(c).$$

(b) To show: $A(b) - A(a) =$ area under $g(x)$ from a to b.

$$A(b) - A(a) = \left(\begin{array}{c}\text{area under } g(x) \\ \text{from } a \text{ to } b\end{array}\right) - \left(\begin{array}{c}\text{area under } g(x) \\ \text{from } a \text{ to } a\end{array}\right)$$

$$= \begin{array}{c}\text{area under } g(x) \\ \text{from } a \text{ to } b.\end{array}$$