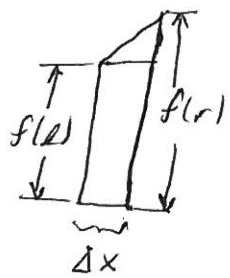
620-295 Real Analysis with applications, lect. 27, 8 October 2009.

Areas

has area $f(l)\Delta x$.

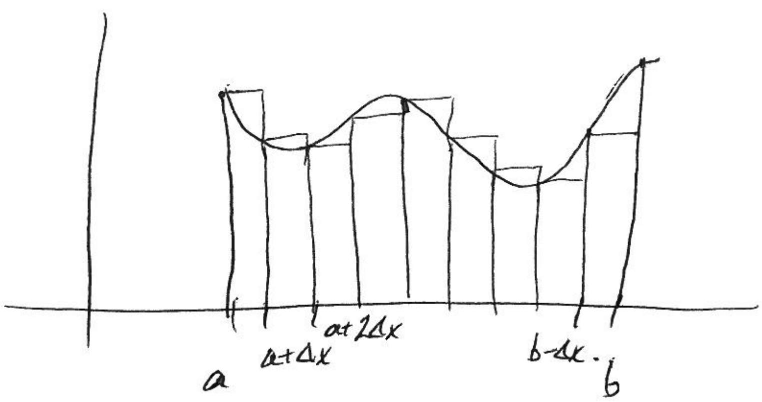


$$\text{has area } f(l)\Delta x + \tfrac{1}{2}\Delta x\,(f(r)-f(l))$$
$$= \frac{\Delta x}{2}\left(2f(l)+f(r)-f(l)\right)$$
$$= \frac{\Delta x}{2}\left(f(l)+f(r)\right).$$

Riemann Integral:

$$\int_a^b f(x)\,dx = \lim_{\Delta x\to 0}\left(\begin{array}{c}\text{sum of the}\\ \text{areas of the}\\ \text{little boxes}\end{array}\right)$$

$$= \lim_{\Delta x\to 0} \Delta x\left(f(a)+f(a+\Delta x)+f(a+2\Delta x)+\cdots + \cdots + f(b-\Delta x)\right)$$



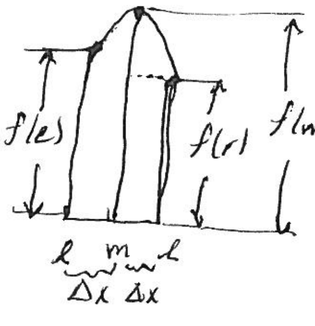
Trapezoidal integral

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \left(\begin{array}{l} \text{sum of the areas of} \\ \text{the little trapezoids} \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{2} \left(\begin{array}{l} f(a) + f(a+\Delta x) + f(a+\Delta x) + f(a+2\Delta x) + \\ + f(a+2\Delta x) + f(a+3\Delta x) + \cdots \\ \cdots + f(b-2\Delta x) + f(b-\Delta x) \\ + f(b-\Delta x) + f(b) \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{2} \left(\begin{array}{l} f(a) + 2f(a+\Delta x) + 2f(a+2\Delta x) + \cdots \\ \cdots + 2f(b-\Delta x) + f(b) \end{array} \right)$$

Simpson's Integral



has area $\frac{\Delta x}{3} \left(f(l) + 4f(m) + f(r) \right)$

so that

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \left(\begin{array}{l} \text{sum of the areas of} \\ \text{the little camel humps} \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{3} \left(\begin{array}{l} f(a) + 4f(a+\Delta x) + f(a+2\Delta x) \\ + f(a+2\Delta x) + 4f(a+3\Delta x) + f(a+4\Delta x) \\ + \cdots \\ + f(b-2\Delta x) + 4f(b-\Delta x) + f(b) \end{array} \right)$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{3} \left(f(a) + 4f(a+\Delta x) + 2f(a+2\Delta x) + 4f(a+3\Delta x) + 2f(a+4\Delta x) + \cdots + f(b) \right)$$

Sequences of functions

Let

$$M([a,b]\to \mathbb{R}) = \{\text{functions } f:[a,b]\to\mathbb{R}\}.$$

Let $(f_1, f_2, f_3, \ldots)$ be a sequence in $M([a,b], \mathbb{R})$.

The sequence (f_n) converges pointwise to $f:[a,b]\to\mathbb{R}$ if (f_n) satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ and $x\in[a,b]$ then there exists $N \in \mathbb{Z}_{>0}$ ~~$N\in\mathbb{R}_{>0}$~~ such that if $n\in\mathbb{Z}_{>0}$ and $n > N$ then $|f_n(x) - f(x)| < \varepsilon$.

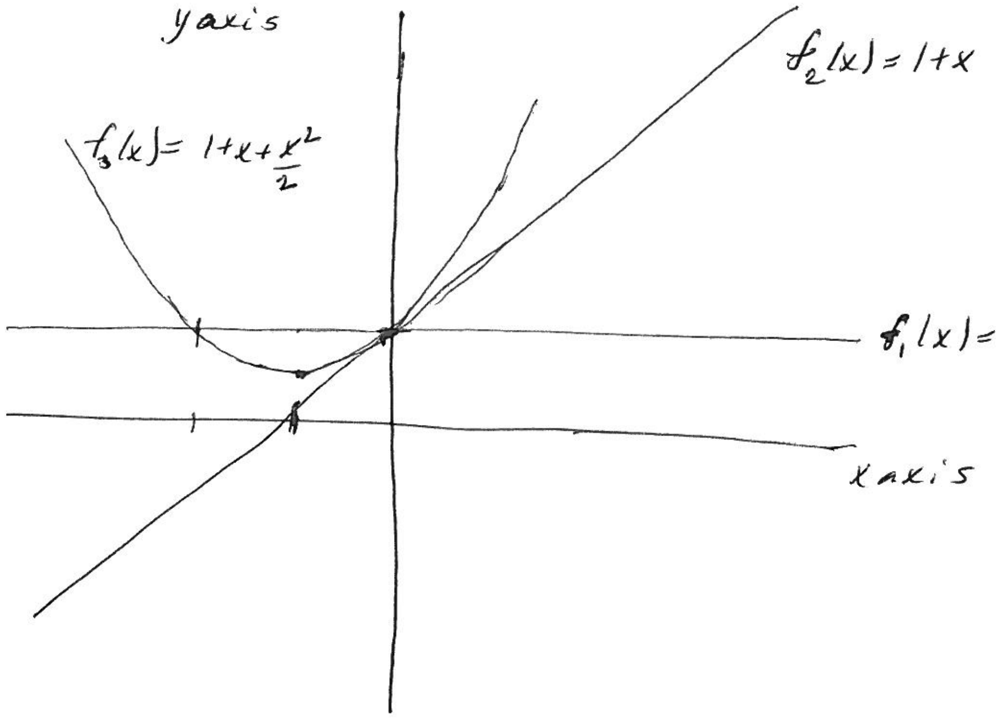
The sequence (f_n) converges uniformly to $f:[a,b]\to\mathbb{R}$ if (f_n) satisfies:

If $\varepsilon\in\mathbb{R}_{>0}$ then there exists $N\in\mathbb{Z}_{>0}$ such that if $x\in[a,b]$ and $n\in\mathbb{Z}_{>0}$ and $n > N$ then $|f_n(x) - f(x)| < \varepsilon$.

Example

$$f_1 = 1,$$
$$f_2 = 1+x,$$
$$f_3 = 1+x+\frac{x^2}{2},$$
$$f_4 = 1+x+\frac{x^2}{2}+\frac{x^3}{6}, \ldots$$

converges to e^x.



$$f_3(x) = 1+x+\frac{x^2}{2} = \frac{1}{2}(2+2x+x^2) = \frac{1}{2}(1+(x+1)^2)$$

Be careful about which functions are being considered.

As functions $f_n : R \to R$, (f_n) does not converge uniformly to $f(x)=e^x$.

As functions $f_n : [0,1] \to R$, (f_n) does converge uniformly to $f(x)=e^x$.

Example Let $f_n : (0,1) \to \mathbb{R}$ be given by

$$f_n(x) = \frac{1}{nx+1}$$

so that

$$f_0(x) = \frac{1}{1} = 1, \quad f_1(x) = \frac{1}{x+1}, \quad f_2(x) = \frac{1}{2x+1}, \ldots$$

This sequence converges to the function $f : (0,1) \to \mathbb{R}$ given by $f(x) = 0$.

It converges pointwise but not uniformly.