# 620-295 Real Analysis with applications 

## Problem Sheet 5

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## 1. Where is a function continuous?

1. For which values of $x$ is the function $f(x)=x^{2}+3 x+4$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
2. For which values of $x$ is the function

$$
f(x)= \begin{cases}\frac{x^{2}-x-6}{x-3}, & \text { if } x \neq 3, \\ 5, & \text { if } x=3,\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
3. For which values of $x$ is the function

$$
f(x)= \begin{cases}\frac{\sin 3 x}{x}, & \text { if } x \neq 0 \\ 1, & \text { if } x=0\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
4. For which values of $x$ is the function

$$
f(x)= \begin{cases}\frac{1-\cos x}{x^{2}}, & \text { if } x \neq 0 \\ 1, & \text { if } x=0\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
5. Determine the value of $k$ for which the function

$$
f(x)= \begin{cases}\frac{\sin 2 x}{5 x}, & \text { if } x \neq 0 \\ k, & \text { if } x=0\end{cases}
$$

continuous at $x=0$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
6. For which values of $x$ is the function

$$
f(x)= \begin{cases}x-1, & \text { if } 1 \leq x<2, \\ 2 x-3, & \text { if } 2 \leq x \leq 3,\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
7. For which values of $x$ is the function

$$
f(x)= \begin{cases}\cos x, & \text { if } x \geq 0 \\ -\cos x, & \text { if } x<0\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
8. For which values of $x$ is the function

$$
f(x)= \begin{cases}\sin (1 / x), & \text { if } x \neq 0 \\ 0, & \text { if } x=0\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
9. Determine the value of $a$ for which the function

$$
f(x)= \begin{cases}a x+5, & \text { if } x \leq 2 \\ x-1, & \text { if } x>2\end{cases}
$$

continuous at $x=2$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
10. For which values of $x$ is the function

$$
f(x)= \begin{cases}1+x^{2}, & \text { if } 0 \leq x \leq 1 \\ 2-x, & \text { if } x>1\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
11. For which values of $x$ is the function $f(x)=2 x-|x|$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
12. Find the value of $a$ for which the function

$$
f(x)= \begin{cases}2 x-1, & \text { if } x<2 \\ a, & \text { if } x=2 \\ x+1, & \text { if } x>2\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
13. For which values of $x$ is the function

$$
f(x)= \begin{cases}\frac{|x-a|}{x-a}, & \text { if } x \neq a \\ 1, & \text { if } x=a\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
14. For which values of $x$ is the function

$$
f(x)= \begin{cases}\frac{x-|x|}{2}, & \text { if } x \neq 0 \\ 2, & \text { if } x=0\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
15. For which values of $x$ is the function

$$
f(x)= \begin{cases}\sin x, & \text { if } x<0 \\ x, & \text { if } x \geq 0\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
16. For which values of $x$ is the function

$$
f(x)= \begin{cases}\frac{x^{n}-1}{x-1}, & \text { if } x \neq 1 \\ \mathrm{n}, & \text { if } x=1\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
17. Explain how you know $f(x)=\cos x$ is continuous for all values of $x$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
18. Explain how you know $f(x)=\cos |x|$ is continuous for all values of $x$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
19. Explain how you know $f(x)=\lfloor x\rfloor$ is continuous for all values of $x$. Justiffy your answer with limits if necessary and draw a graph of the function to illustrate your answer.
20. For what values of $x$ is the function

$$
f(x)= \begin{cases}x^{3}-x^{2}+2 x-2, & \text { if } x \neq 1 \\ 4, & \text { if } x=1\end{cases}
$$

continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.
21. For what values of $x$ is the function $f(x)=|x|+|x-1|,-1 \leq x \leq 2$, continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

## 2. Fundamental theorem of Calculus

1. What does $\int_{a}^{b} f(x) d x$ mean?
2. How does one usually calculate $\int_{a}^{b} f(x) d x$ ? Give an example which shows that this method does not always work. Why doesn't it?
3. Give an example which shows that $\int_{a}^{b} f(x) d x$ is not always the true area under $f(x)$ between $a$ and $b$ even if $f(x)$ is continuous between $a$ and $b$.
4. What is the Fundamental Theorem of Calculus?
5. Let $f(x)$ be a function which is continuous and let $A(x)$ be the area under $f(x)$ from $a$ to $x$. Compute the derivative of $A(x)$ by using limits.
6. Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.
7. Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this, compute an area by summing up the areas of tiny boxes and then show that applying the Fundamental Theorem of Calculus gives the same result.

## 3. Improper integrals

1. Show that if $z=\sqrt{\frac{4+x}{1-x}}$ then $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} d x=\int_{0}^{\infty} \frac{4 z^{2}}{\left(z^{2}+1\right)^{2}} d z$. The improper integral on the left is an improper integral of the first kind and the improper integral on the right is an improper integral of the second kind.
2. Show that $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} d x=\pi$.
3. Show that $\int_{0}^{1} \frac{1}{x} d x$ diverges.
4. Evaluate $\int_{0}^{3} \frac{d x}{(x-1)^{2 / 3}}$.
5. Determine whether $\int_{1}^{\infty} \frac{1}{x} d x$ converges or diverges.
6. Determine whether $\int_{1}^{\infty} \frac{1}{x^{2}} d x$ converges or diverges.
7. Determine whether $\int_{1}^{\infty} e^{-x^{2}} d x$ converges or diverges.
8. Show that $\int_{1}^{\infty} \frac{1}{x^{p}} d x$ converges if $p \in \mathbb{R}$ and $p>1$.
9. Show that $\int_{1}^{\infty} \frac{1}{x^{p}} d x$ diverges if $p \in \mathbb{R}$ and $p \leq 1$.
10. Evaluate $\int_{0}^{\infty} \frac{1}{x^{2}+1} d x$.
11. Evaluate $\int_{0}^{1} \frac{1}{\sqrt{x}} d x$.
12. Evaluate $\int_{-1}^{1} \frac{1}{x^{2 / 3}} d x$.
13. Evaluate $\int_{1}^{\infty} \frac{1}{x^{1.001}} d x$.
14. Evaluate $\int_{0}^{4} \frac{1}{\sqrt{4-x}} d x$.
15. Evaluate $\int_{0}^{1} \frac{1}{\sqrt{1-x^{2}}} d x$.
16. Evaluate $\int_{0}^{\infty} e^{-x} \cos x d x$.
17. Evaluate $\int_{0}^{1} \frac{1}{x^{0.999}} d x$.
18. Determine whether $\int_{1}^{\infty} \frac{1}{\sqrt{x}} d x$ converges or diverges.
19. Determine whether $\int_{1}^{\infty} \frac{1}{x^{3}} d x$ converges or diverges.
20. Determine whether $\int_{1}^{\infty} \frac{1}{x^{3}+1} d x$ converges or diverges.
21. Determine whether $\int_{0}^{\infty} \frac{1}{x^{3}} d x$ converges or diverges.
22. Determine whether $\int_{0}^{\infty} \frac{1}{x^{3}+1} d x$ converges or diverges.
23. Determine whether $\int_{0}^{\infty} \frac{1}{1+e^{x}} d x$ converges or diverges.
24. Determine whether $\int_{0}^{\pi / 2} \tan x d x$ converges or diverges.
25. Determine whether $\int_{-1}^{1} \frac{1}{x^{2}} d x$ converges or diverges.
26. Determine whether $\int_{-1}^{1} \frac{1}{x^{2 / 5}} d x$ converges or diverges.
27. Determine whether $\int_{0}^{\infty} \frac{1}{\sqrt{x}} d x$ converges or diverges.
28. Determine whether $\int_{0}^{\infty} \frac{1}{\sqrt{x+x^{4}}} d x$ converges or diverges.
29. Classify the following improper integrals and evaluate them if they converge:
(i) $\int_{1}^{5} \frac{4 x}{\sqrt{x^{2}-1}}$.
(ii) $\int_{1}^{\infty} \frac{1}{1+x^{2}}$.
(iii) Does the following integral diverge or converge? Explain why, but do not evaluate the integral. $\int_{1}^{\infty} \frac{x^{2}}{(x-2)\left(x^{11}+2\right)^{1 / 4}}$.
