620-295 Real Analysis with applications Lect. 28, 12.10.2009 

Error estimates for approximations

The trapezoidal integral If $f:[a,b]\to\mathbb{R}$ then

$$\int_a^b f(x)\,dx = \lim_{\Delta x\to 0} \frac{\Delta x}{2}\left(f(a)+2f(a+\Delta x)+\cdots+2f(b-\Delta x)+f(b)\right)$$

(if it exists).

Say $\Delta x = \frac{b-a}{N}$ (we divide $[a,b]$ into $N$ parts)

Then

$$\mathrm{Error}(N) = \int_a^b f(x)dx - \left(\frac{b-a}{N}\right)\frac{1}{2}\Big(f(a)+2f(a+\Delta x)+\cdots \cdots + 2f(b-\Delta x)+f(b)\Big)$$

Theorem

Assume that the second derivative $f'':[a,b]\to\mathbb{R}$ exists, is continuous and

$$|f''(x)| < M, \text{ for } x\in[a,b].$$

Then

$$|\mathrm{Error}(N)| < \frac{(b-a)^3}{12N^2}\cdot M$$

Example Find $\log(2)$ to within .01

$$\log(2) = \int_1^2 \frac{1}{x}\,dx.$$

So let $f(x) = \frac{1}{x}$, $f : [1,2] \to \mathbb{R}$.

Then $f'' : [1,2] \to \mathbb{R}$ is given by $\frac{2}{x^3} = f''(x)$

So $|f''(x)| \leq 2$ for $x \in [1,2]$ and

$$|\text{Error}(N)| \leq \frac{(2-1)^3}{12 \cdot N^2} \cdot 2 = \frac{1}{6N^2}.$$

So, if $N = 5$ then $|\text{Error}(N)| \leq \frac{1}{6 \cdot 25} = \frac{1}{150} < .01$.

So $\log(2)$ is approximately equal to

$$\frac{(2-1)}{5} \cdot \frac{1}{2} \left( \frac{1}{1} + 2\frac{1}{1+\frac{1}{5}} + 2\frac{1}{1+\frac{2}{5}} + 2\frac{1}{1+\frac{3}{5}} + 2\frac{1}{1+\frac{4}{5}} + \frac{1}{2} \right)$$

$$= \frac{1}{10}\left(1 + \frac{2 \cdot 5}{6} + \frac{2 \cdot 5}{7} + \frac{2 \cdot 5}{8} + \frac{2 \cdot 5}{9} + \frac{1}{2}\right)$$

$$= \frac{1}{10} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{20}.$$

<u>Taylor's Theorem</u> If $f : [a,b] \to \mathbb{R}$ then

$$f(a+x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

with

$$a_\ell = \frac{1}{\ell!}\left( \frac{d^\ell}{dx^\ell} f \Big|_{x=a} \right) = \frac{1}{\ell!} f^{(\ell)}(a).$$

Theorem

Assume $f^{(N-1)}:[a,b]\to\mathbb{R}$ is continuous and $f^{N}(x)$ exists for $x\in[a,b]$. Then there exists $c\in[a,a+x]$ such that

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_{N-1}x^{N-1}+\text{Error}(N)$$

where

$$a_\ell=\frac{1}{\ell!}f^{(\ell)}(a) \quad\text{and}\quad \text{Error}(N)=\frac{1}{N!}f^{(N)}(c)\,x^N.$$

Example Find $\log(2)$ to within $.01$

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{x^{N-1}}{N-1}+\text{Error}(N)$$

where $\text{Error}(N)=??$.

Since

$$\frac{d}{dx}\log(1+x)=\frac{1}{1+x},\quad \frac{d^2}{dx^2}\log(1+x)=\frac{-1}{(1+x)^2},$$

$$\frac{d^3}{dx^3}\log(1+x)=\frac{2}{(1+x)^3},\ \ldots,\ \frac{d^N}{dx^N}\log(1+x)=\frac{(-1)^{N-1}(N-1)!}{(1+x)^N}$$

then

$$\text{Error}(N)=\frac{1}{N!}\frac{(-1)^{N-1}(N-1)!}{(1+c)^N}x^N \quad\text{with } c\in[1,x].$$

So

$$|\text{Error}(N)|\leq\frac{1}{N}\frac{1^n}{1^N}=\frac{1}{N}.$$

Thus, within .01

$$\log(2) \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99}.$$

<u>Example</u> Approximate $\sqrt{17}$.

Since $\sqrt{17} = \sqrt{16+1} = (16+x)^{\frac{1}{2}}\Big|_{x=1}$, and

$$\frac{d}{dx}(16+x)^{\frac{1}{2}} = \frac{\frac{1}{2}}{(16+x)^{\frac{1}{2}}}, \quad \frac{d^2}{dx^2}(16+x)^{\frac{1}{2}} = \frac{\frac{1}{2}(-\frac{1}{2})}{(16+x)^{3/2}}, \quad \frac{d^3}{dx^3}(16+x)^{\frac{1}{2}} = \frac{\frac{1}{2}(\frac{-1}{2})\frac{3}{2}}{(16+x)^{5/2}},$$

then

$$(16+x)^{\frac{1}{2}} = 16^{\frac{1}{2}} + \frac{\frac{1}{2}}{16^{\frac{1}{2}}}x + \frac{1}{2!}\frac{\frac{1}{2}(-\frac{1}{2})}{16^{3/2}}x^2 + \frac{1}{3!}\frac{\frac{1}{2}(\frac{-1}{2})(\frac{3}{2})}{16^{5/2}}x^3 + \cdots$$

and

$$\sqrt{17} = (16+1)^{\frac{1}{2}} = 4 + \frac{1}{2 \cdot 4} - \frac{1}{2 \cdot 2 \cdot 2 \cdot 4^3} + \frac{3}{2 \cdot 3 \cdot 2^3 \cdot 4^5} - \cdots$$

$$= 4 + \frac{1}{8} - \frac{1}{2^9} + \frac{1}{2^{14}} - \cdots$$

$$= 4 + \frac{1}{8} - \frac{1}{512} + \frac{1}{2^{14}} - \cdots$$

Since $\frac{1}{2^{14}} = \left(\frac{1}{128}\right)^2 < \frac{1}{100^2} = 10^{-4} = .0001$,

with .0001,

$$\sqrt{17} \approx 4 + \frac{1}{8} - \frac{1}{512} + \frac{1}{2^{14}}.$$

Remark: If $N = 1$ then Taylor's theorem says

If $f'(x)$ exists for $x \in [a,b]$ then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + Error(1) = f(a) + f'(c)(b-a)$$

This is just the mean value theorem.

So Taylor's theorem is the mean value theorem generalized to all $N$. (in fact, the proof of Taylor's theorem is by induction on $N$).