620-295 Real Analysis with applications, Lect. 29, 15.10.2009

Limits

$l = \lim_{x \to a} f(x)$ means:

If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $d(x,a) < \delta$ then $d(f(x), l) < \varepsilon$.

$l = \lim_{n \to \infty} a_n$ means:

If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $d(a_n, l) < \varepsilon$.

Theorems Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

Then

(a) $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$,

(b) If c is a constant then

$$\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x)$$

(c) $\lim_{x \to a} f(x) g(x) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right)$

(d) f is continuous at $x = a$ if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Useful limits

(a) If $n \in \mathbb{Z}_{>0}$ then $\lim\limits_{x\to\infty} x^n e^{-x} = 0$.

(b) If $\alpha \in \mathbb{R}_{>0}$ then $\lim\limits_{x\to\infty} x^{-\alpha} \log x = 0$.

(c) Let $p \in \mathbb{R}_{>0}$. Then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$.

(d) Let $p \in \mathbb{R}_{>0}$. Then $\lim\limits_{n\to\infty} p^{\frac{1}{n}} = 1$.

(e) $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

(f) Let $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}_{>0}$. Then $\lim\limits_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

(g) If $|x|<1$ then $\lim\limits_{n\to\infty} x^n = 0$.

(h) ~~If~~ $0<\alpha<1$ then $\lim\limits_{x\to 0} x^\alpha = 0$

(i) $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$

(j) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

(k) $\lim\limits_{x\to 0} \frac{\cos x - 1}{x^2} = \frac{-1}{2}$

(l) $\lim\limits_{x\to 0} \frac{\log(1+x)}{x} = 1$.

Proof of (l)

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{y\to 0}\frac{\log(1+e^y-1)}{e^y-1}$$

$$= \lim_{y\to 0}\frac{y}{e^y-1} = \lim_{y\to 0}\frac{1}{\frac{e^y-1}{y}} = \frac{1}{1} = 1.$$

Proof of (j)

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{e^{ix}-e^{-ix}}{2ix} = \lim_{x\to 0}\frac{(e^{ix}-1)}{2ix} - \frac{(e^{-ix}-1)}{2ix}$$

$$= \lim_{x\to 0}\frac{1}{2}\left(\frac{e^{ix}-1}{ix}\right) + \frac{1}{2}\left(\frac{e^{-ix}-1}{-ix}\right) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1.$$

Proof of (a) Assume $n \in \mathbb{Z}_{>0}$

$$0 \le \lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty}\frac{x^n}{e^x}$$

$$\le \lim_{x\to\infty}\frac{x^n}{\frac{1}{(n+1)!}x^{n+1}}$$

$$= \lim_{x\to\infty}\frac{(n+1)!}{x} = (n+1)!\lim_{x\to\infty}\frac{1}{x} = 0$$

Proof of (b) Let $0 < \varepsilon < \alpha$.

$$0 \le \lim_{x\to\infty} x^{-\alpha}\log x = \lim_{x\to\infty}\left(x^{-\alpha}\int_1^x \frac{1}{t}\,dt\right)$$

$$\le \lim_{x\to\infty}\left(x^{-\alpha}\int_1^x t^{\varepsilon-1}\,dt\right)$$

$$= \lim_{x\to\infty} x^{-\alpha}\left(\frac{x^{\varepsilon}-1^{\varepsilon}}{\varepsilon}\right) = \lim_{x\to\infty}\left(\frac{x^{\varepsilon-\alpha}-x^{-\alpha}}{\varepsilon}\right)$$

$$\le \lim_{x\to\infty}\frac{x^{\varepsilon-\alpha}}{\varepsilon} = 0.$$

Proof of (c) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{>0}$ and $n > N$ then $\left|\frac{1}{n^p}\right| < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

Let $N = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$.

To show: If $n \in \mathbb{Z}_{>0}$ and $n > N$ then $\left|\frac{1}{n^p}\right| < \varepsilon$

Assume $n \in \mathbb{Z}_{>0}$ and $n > N$.

To show: $\left|\frac{1}{n^p}\right| < \varepsilon$

$$\left|\frac{1}{n^p}\right| < \left|\frac{1}{N^p}\right| = \frac{1}{\left(\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\right)^p} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

Proof of (d) $\lim_{n\to\infty} p^{\frac{1}{n}} = \lim_{n\to\infty} \left(e^{\log p}\right)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log p}$

$$= e^{\lim_{n\to\infty} \frac{1}{n}\log p} = e^{(\log p)\lim_{n\to\infty}\frac{1}{n}} = e^{\log p \cdot 0}$$

$$= e^{0} = \underline{1}.$$

Proof of (e) ~~Assum~~ To show: $\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$

To show: $\lim_{n\to\infty} \left(n^{\frac{1}{n}} - 1\right) = 0.$

We know:

$$n = \left(n^{\frac{1}{n}}\right)^n = \left(\left(n^{\frac{1}{n}}-1\right)+1\right)^n = 1 + n\left(n^{\frac{1}{n}}-1\right) + \frac{n(n-1)}{2}\left(n^{\frac{1}{n}}-1\right)^2 + \cdots$$

$$\geq \frac{n(n-1)}{2}\left(n^{\frac{1}{n}}-1\right)^2.$$

So $\left(n^{\frac{1}{n}}-1\right)^2 \leq \frac{2}{n-1}.$

So
$$0 \leq \lim_{n\to\infty}\left(n^{\frac{1}{n}}-1\right)^2 \leq \lim_{n\to\infty}\frac{2}{n-1} = 0$$

So $\lim_{n\to\infty}\left(n^{\frac{1}{n}}-1\right) = 0.$