620-295 Real Analysis with applications, Lect 32, 20.09.2009

Taylor series

Theorem

(a) If $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$ then

$$a_\ell = \frac{1}{\ell!}\left(\frac{d^\ell f}{dx^\ell}\Big|_{x=0}\right)$$

Remark: A tiny bit of algebra gives

If $f(a+x) = a_0 + a_1 x + a_2 x^2 + \dots$ then $a_\ell = \frac{1}{\ell!}\left(\frac{d^\ell f}{dx^\ell}\Big|_{x=0}\right)$

and

If $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$ then $a_\ell = \frac{1}{\ell!}\left(\frac{d^\ell f}{dz^\ell}\Big|_{z=a}\right)$

(b) If $f : [a,b] \to \mathbb{R}$ and $N \in \mathbb{Z}_{>0}$ and $f^{(N)} : [a,b] \to \mathbb{R}$ is continuous and $f^{(N+1)} : (a,b) \to \mathbb{R}$ exists then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!}f''(a)(b-a)^2 + \dots + \frac{1}{N!}f^{(N)}(a)(b-a)^N$$
$$+ \frac{1}{(N+1)!}f^{(N+1)}(c)(b-a)^{N+1}$$

Remark: The special case $N=1$ is the Mean Value Theorem

The special case $N=1$ and $f(a) = f(b)$ is Rolle's Theorem

The last term in $f(b) = f(a) + f'(a)(b-a) + \dots + \frac{1}{(N+1)!}f^{(N+1)}(c)(b-a)^{N+1}$ is Lagrange's form of the remainder

Examples of Taylor series

$\frac{1}{1-x}$, $\frac{1}{1+x}$, $\frac{1}{1+x^2}$, $\arctan x$, $\log(1+x)$

e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$,

$(1+x)^{1/3}$, $(1+x)^{\frac{1}{2}}$

Recall: $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$, $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x+e^{-x}}{2}$

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

Examples of MVT problems

Examples of Taylor's theorem problems

(1) Explain what Picard iteration is

(2) Explain what Newton iteration is.

Proof of the mean value theorem and Taylor's theorem (3)

(a) Assume $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

To show: $a_\ell = \frac{1}{\ell!}\left(\frac{d^\ell f}{dx^\ell}\Big|_{x=0}\right)$ for $\ell \in \mathbb{Z}_{\geq 0}$.

Proof by induction.

<u>Case $\ell=0$</u> $a_0 = f\big|_{x=0} = f(0)$.

<u>Case $\ell=1$</u> To show: $a_1 = \frac{df}{dx}\Big|_{x=0}$.

$\frac{df}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$. So $\frac{df}{dx}\Big|_{x=0} = a_1$.

<u>Case $\ell=2$</u> To show: $a_2 = \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\Big|_{x=0}\right)$.

$\frac{d^2 f}{dx^2} = 2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + \cdots$. So $\frac{d^2 f}{dx^2}\Big|_{x=0} = 2a_2$.

So $a_2 = \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\Big|_{x=0}\right)$.

<u>Induction step:</u> Assume $a_r = \frac{1}{r!}\left(\frac{d^r f}{dx^r}\Big|_{x=0}\right)$ for $r < \ell$.

To show: $a_\ell = \frac{1}{\ell!}\left(\frac{d^\ell f}{dx^\ell}\Big|_{x=0}\right)$.

The point really is:

$$\frac{d^\ell f}{d\ell} = \ell(\ell-1)\cdots 2\cdot 1\, a_\ell + (\ell+1)\ell(\ell-1)\cdots 2\, a_{\ell+1} x +$$
$$+ (\ell+2)(\ell+1)\cdots 4\cdot 3\, a_{\ell+2} x^2 + \cdots \quad \text{so that}$$

$$\left.\frac{d^{\ell} f}{dx^{\ell}}\right|_{x=0} = \ell!\, a_{\ell} \text{ and } a_{\ell} = \frac{1}{\ell!}\left(\left.\frac{d^{\ell} f}{dx^{\ell}}\right|_{x=0}\right)$$

<u>Proof of the Mean Value theorem</u>

To show: There exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(c)(b-a).$$

First do the <u>case</u>: $f(a) = f(b)$.

Then, since $[a,b]$ is compact and connected and $f: [a,b] \to \mathbb{R}$ is cont.

$f([a,b])$ is compact and connected.

So $f([a,b])$ is a closed interval.

So $f([a,b]) = [\min, \max]$ for some $\min, \max \in \mathbb{R}$.

So there exists $c \in (a,b)$ such that $f(c) = \max$.

Then, if $\varepsilon \in \mathbb{R}_{>0}$ is small enough

$f(c+\varepsilon) \leq f(c)$ and $f(c-\varepsilon) \leq f(c)$.

So $f'(c) \geq 0$ and $f'(c) \leq 0$.

So $f'(c) = 0$.

Next do the <u>case</u> $f(a) \neq f(b)$

Let

$$g(x) = -\left(\frac{f(b)-f(a)}{b-a}\right)(x-a) + f(x).$$

so that $g(a) = f(a)$ and $g(b) = f(a)$ and $g'(x) = -\left(\frac{f(b)-f(a)}{b-a}\right) + f'(x)$

So there exists $c \in (a,b)$ with $g'(c) = 0$.

So
$$f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right) = 0.$$

So $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ and $f(b) = f(a) + f'(c)(b-a)$.