Topology

A topological space is a set X with a topology $\mathcal{T}$.

Let X be a set.

A topology on X is a collection of subsets of $\mathcal{T}$ such that

(a) ~~Any~~ union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

(b) Any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

(c) $\emptyset$ is an element of $\mathcal{T}$ and X is an element of $\mathcal{T}$

Let X be a topological space. Let $E \subseteq X$.

The set E is open if E is an element of $\mathcal{T}$

The set E is closed if E^c is open.

The set E is connected if there do not exist open sets A and B with

$$A \cap E \neq \emptyset,\ B \cap E \neq \emptyset,\ (A \cup B) \cap E = E$$

and $(A \cap B) \cap E = \emptyset$

The set E is compact if E satisfies:

every open cover of E has a finite subcover

Let X and Y be topological spaces.

A continuous function from X to Y is a function $f: X \to Y$ such that

if V is an open set of Y then

$f^{-1}(V)$ is an open set of X.

Theorems

(a) The interior of E is the set of interior points of E

(b) The closure of E is the set of close points of E.

(c) If $f: X \to Y$ is a continuous function and X is connected then $f(X)$ is connected.

(d) If $f: X \to Y$ is a continuous function and X is compact then $f(X)$ is compact.

Examples of topological spaces

spheres, tori, Klein bottles, Riemann surfaces...

and $\mathbb{R}^n$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{C}$.

$\mathbb{R}^n$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ are all metric spaces.

A <u>metric space</u> is a set X with a function $d: X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $p \in X$ then $d(p,p) = 0$,

(b) If $p, q \in X$ and $p \neq q$ then $d(p,q) \neq 0$,

(c) If $p, q \in X$ then $d(p,q) = d(q,p)$,

(d) If $p, q, r \in X$ then $d(p,r) \leq d(p,q) + d(q,r)$.

Let X be a metric space. Let $\varepsilon \in \mathbb{R}_{>0}$ and $p \in X$. The <u>ε-ball</u> at p is

$$B_\varepsilon(p) = \{ q \in X \mid d(q,p) < \varepsilon \}$$

Define $\mathbb{R}^n = \{ (x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R} \}$.

<u>Theorem</u>

(a) Let X be a metric space and let

$$\mathcal{T} = \{ E \subseteq X \mid E \text{ is a union of } \varepsilon\text{-balls} \}$$

Then X (with $\mathcal{T}$) is a topological space

(b) Let $n \in \mathbb{Z}_{>0}$. Let $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be given by

$$d(q,p) = \sqrt{(q_1-p_1)^2+(q_2-p_2)^2+\cdots+(q_n-p_n)^2}$$

Then $\mathbb{R}^n$ (with d) is a metric space.

<u>Theorem</u>

(a) Let $E \subseteq \mathbb{R}$. The set E is connected if and only if E is an interval.

(b) Let $E \subseteq \mathbb{R}$. The set E is connected and compact if and only there exist $a, b \in \mathbb{R}$ such that $E = [a,b]$.

(c) Let $E \subseteq \mathbb{R}$. The set E is compact if and only if E is closed and bounded.