Numbers

The little box:

$$\mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}.$$

$\mathbb{Z}_{>0}$, with operation $+: \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, is the monoid possibly without identity generated by 1,

$$\mathbb{Z}_{>0} = \{1, 1+1, 1+1+1, 1+1+1+1, \ldots\}$$

$$\mathbb{Z}_{\geq 0} = \{0, 1, 1+1, 1+1+1, \ldots\}$$ monoid generated by 1

$$= \{0, 1, 2, 3, \ldots\}$$

$$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, 3, \ldots\}$$ group generated by 1.

In a group G with operation $+: G \times G \to G$, $(a,b) \mapsto a+b$

$-a$ means "the element of G such that when you add it to a you get the identity".

$\mathbb{Q}$ is constructed from $\mathbb{Z}$: $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ with

$$\frac{a}{b} = \frac{c}{d} \quad \text{if} \quad ad = bc,$$

and

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

A *field* is a set $\mathbb{F}$ with operations

$$+: \mathbb{F} \times \mathbb{F} \to \mathbb{F} \quad \text{and} \quad \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$
$$(a,b) \mapsto a+b \qquad\qquad (a,b) \mapsto ab$$

such that

(a) If $a,b,c \in \mathbb{F}$ then $(a+b)+c = a+(b+c)$,

(b) If $a,b \in \mathbb{F}$ then $a+b = b+a$,

(c) There exists $0 \in \mathbb{F}$ such that if $a \in \mathbb{F}$ then $0+a = a$ and $a+0 = a$,

(d) If $a \in \mathbb{F}$ then there exists $-a \in \mathbb{F}$ such that
$$a+(-a) = 0 \text{ and } (-a)+a = 0$$

(e) If $a,b,c \in \mathbb{F}$ then $(ab)c = a(bc)$.

(d) If $a,b,c \in \mathbb{F}$ then
$$(a+b)c = ac+bc \text{ and } c(a+b) = ca+cb$$

(e) ~~If $a \in \mathbb{F}$ then~~ There exists $1 \in \mathbb{F}$ such that if $a \in \mathbb{F}$ then $1a = a1 = a$,

(f) If $a \in \mathbb{F}$ and $a \neq 0$ then there exists $a^{-1} \in \mathbb{F}$ such that
$$aa^{-1} = 1 \text{ and } a^{-1}a = 1.$$

All the functions

$$(*) \qquad \mathbb{Z}_{>0} \longrightarrow \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R} \longrightarrow \mathbb{C}$$

are injective but not surjective:

$$0 \notin \mathbb{Z}_{>0}, \quad -1 \notin \mathbb{Z}_{\geq 0}, \quad \tfrac{1}{2} \notin \mathbb{Z}, \quad \sqrt{2} \notin \mathbb{Q}, \quad i \notin \mathbb{R}.$$

The map $\mathbb{Q} \longrightarrow \mathbb{R}$ is the <u>long division function</u>.

All the functions

$$\mathbb{Z}_{>0} \xrightarrow{f_1} \mathbb{Z}_{\geq 0} \xrightarrow{f_2} \mathbb{Z} \xrightarrow{f_3} \mathbb{Q} \xrightarrow{f_4} \mathbb{R} \xrightarrow{f_5} \mathbb{C}$$

satisfy

(a) $f_i(x+y) = f_i(x) + f_i(y)$

(b) $f_i(1) = 1$

(c) $f_i(xy) = f_i(x) f_i(y)$

<u>Example of long division:</u>

$$\frac{13}{7} = 1.857142857142857142\ldots$$

since

$$\begin{array}{r}
1.857142 \\
7\,\overline{)\,13.00000000} \\
\underline{7} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
10 \\
\underline{7} \\
30 \\
28
\end{array}$$

Why is it a repeating decimal?

Order

The order on $\mathbb{R}$ is the relation $\le$ given by

$$a \le b \quad \text{if} \quad b - a \in \mathbb{R}_{\ge 0},$$

where $\mathbb{R}_{\ge 0} = \{ a_0 . a_1 a_2 a_3 \ldots \mid a_0 \in \mathbb{Z}_{\ge 0},\ a_1, a_2, \ldots \in \{0,1,\ldots,9\} \}$.

Let S be a set.

A partial order on S is a relation $\le$ on S such that

(a) if $x, y, z \in S$ and $x \le y$ and $y \le z$ then $x \le z$,

(b) if $x, y \in S$ and $x \le y$ and $y \le x$ then $x = y$.

A total order on S is a partial order on S such that

if $x, y \in S$ then either $x \le y$ or $y \le x$.

An ordered field is a field $\mathbb{F}$ with a total order $\le$ such that

(a) if $x, y, z \in \mathbb{F}$ and $x \le y$ then $x + z \le y + z$,

(b) if $x, y, z \in \mathbb{F}$ and $x \ge 0$ and $y \ge 0$ then $xy \ge 0$.

Theorem $\mathbb{R}$ with operations $\begin{matrix} + : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (a,b) \mapsto a+b \end{matrix}$ and $\begin{matrix} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (a,b) \mapsto ab \end{matrix}$ and order $\le$ on $\mathbb{R}$ is an ordered field.

<u>Proposition</u> Let $\mathbb{F}$ be an ordered field with order $\leq$. Then

(a) if $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$,

(b) if $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$,

(c) if $a, b \in \mathbb{F}$ and $a > 0$ and $b > 0$ then $ab > 0$.

(d) If $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$ then

$a \leq b$ if and only if $a^2 \leq b^2$.