620-295 Real Analysis with applications Lect. 35, 27.10.2009 ①

Sequences and Series

A sequence in $\mathbb{R}$ is a function $\mathbb{Z}_{>0} \to \mathbb{R}$
$n \mapsto a_n$

$\lim_{n\to\infty} a_n = l$ means: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n > N$ then $|a_n - l| < \varepsilon$.

$\sum_{n\to\infty} a_n = s$ means: $\lim_{k\to\infty} s_k = s$, where

$s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \ldots$

Think about these in terms of the graph of (a_n).

Examples of sequences

(a) $a_n = \frac{1}{n}$

(b) $a_n = n$

(c) $a_n = (-1)^n$

(d) $a_n = x^n$

(e) $a_n = \left(1 + \frac{r}{n}\right)^n$

(f) $a_n = (-1)^n\left(1 + \frac{1}{n}\right)$

(g) $a_n = n^{1/n}$

(h) $a_n = \frac{x^n}{n!}$

(i) $a_n = \frac{\log n}{n}$

(j) $a_n = x^{1/n}$

(k) Picard: $a_{n+1} = f(a_n)$
Then $f(l) = l$

(l) Newton: $a_{n+1} = \frac{-f(a_n)}{f'(a_n)} + a_n$
Then $f(l) = 0$

Examples of Series

(a) $\sum_{n=1}^{\infty} x^n$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (c) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$.

Power series: e^x, $\sin x$, $\sinh x$, $\frac{1}{1-x}$, $\frac{1}{1+x^2}$, $\arctan x$, $\log(1+x)$.

Definitions

(a) upper bound

(b) bounded above

(c) sup and inf

(d) least upper bound
greatest lower bound

(e) limsup and liminf

(f) increasing
decreasing
monotone

(g) Cauchy

(h) conditionally convergent
absolutely convergent.

(i) contractive

The sequence $a_n = (-1)^n(1+\frac{1}{n})$ gives good examples for sup, inf, limsup, liminf.

The sequence $\frac{\log(n)}{n}$ gives good examples for increasing, decreasing, monotone, bounded.

The ~~sequence~~ series $\sum_{n=1} (-1)^n \frac{1}{n}$ is a standard example of a conditionally convergent series.

<u>Leibniz's theorem</u> If (a_n) is a sequence in $\mathbb{R}$ such that

(a) $a_n \in \mathbb{R}_{\geq 0}$,

(b) if $n \in \mathbb{Z}_{>0}$ then $a_n \geq a_{n+1}$,

(c) $\lim_{n\to\infty} a_n = 0$,

then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

<u>Proof</u> Assume (a_n) is a sequence in $\mathbb{R}$ and $a_n \in \mathbb{R}_{\geq 0}$, if $n \in \mathbb{Z}_{>0}$ then $a_n \geq a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$.

To show: $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Let
$$s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}).$$

Then $s_{2m} \leq s_{2(m+1)}$.

Since $s_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$,

$$s_{2m} \leq a_m$$

So the sequence $(s_2, s_4, s_6, \ldots)$ is increasing and bounded above. So $\lim_{n\to\infty} s_{2m}$ exists.

Let $l = \lim_{m\to\infty} s_{2m}$.

Since $s_{2m+1} = s_{2m} + a_{2m+1}$, $\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + \lim_{m\to\infty} a_{2m+1}$

$$= l + 0 = l.$$

So $\lim_{m\to\infty} s_m = l$. //

A sequence (a_n) is <u>contractive</u> if there exists $\alpha \in \mathbb{R}$, $\alpha \in (0,1)$ such that

$$|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}|, \text{ for } n = 2, 3, 4, \ldots$$

If (a_n) is contractive then

$$\begin{aligned} |a_{n+1} - a_n| &\leq \alpha |a_n - a_{n-1}| \\ &\leq \alpha^2 |a_{n-1} - a_{n-2}| \\ &\leq \alpha^3 |a_{n-2} - a_{n-3}| \\ &\ \ \vdots \\ &\leq \alpha^{n-1} |a_{n-(n-2)} - a_{n-(n-1)}| \\ &\leq \alpha^{n-1} |a_2 - a_1|. \end{aligned}$$

This is the idea behind the proof of the ratio test.

Example for which ratio test or root test. is useful:

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^n (x-2)^n}{n\, 3^n}$$

<u>Theorem</u> Let (a_n) be a sequence in $\mathbb{R}$.

(a) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ then $\sum_{n \to \infty} |a_n|$ converges

(b) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ then $\sum_{n=1}^{\infty} |a_n|$ diverges.

<u>Proof</u>

(a) Assume $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l$ and $l<1$.

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $l < l+\varepsilon < 1$.

Let $N \in \mathbb{Z}_{>0}$ such that if $n>N$ then

$$\left|\frac{|a_{n+1}|}{|a_n|} - l\right| < \varepsilon.$$

Then

$$\sum_{n=1}^{\infty}|a_n| = |a_1|+|a_2|+\cdots+|a_N|+|a_{N+1}|+|a_{N+2}|+\cdots$$

$$=|a_1|+\cdots+|a_N|+\frac{|a_{N+1}|}{|a_N|}|a_N| + \frac{|a_{N+2}|}{|a_{N+1}|}\frac{|a_{N+1}|}{|a_{N+2}|}|a_N|+\cdots$$

$$\leq |a_1|+\cdots+|a_{N-1}|+|a_N|\left(1+(l+\varepsilon)+(l+\varepsilon)^2+\cdots\right)$$

$$\leq |a_1|+\cdots+|a_{N-1}|+|a_N|+\frac{1}{1-(l+\varepsilon)}$$

So $\sum_{n=1}^{\infty}|a_n|$ is a <u>comparison</u> to a geometric series!

Assume $\lim_{n\to\infty} |a_n|^{1/n} = l$ and $l<1$.
Let $\varepsilon \in \mathbb{R}_{>0}$ s.t. $l < l+\varepsilon < 1$.
Let N be s.t. if $n > N$ then $\left||a_n|^{1/n} - l\right| < \varepsilon$.

Then
$$\sum_{n=1}^{\infty} |a_n| = |a_1| + \cdots + |a_{N-1}| + |a_N| + |a_{N+1}| + \cdots$$

$$= |a_1| + \cdots |a_{N-1}| + |a_N| + \left(|a_{N+1}|^{1/N+1}\right)^{N+1} + \left(|a_{N+2}|^{1/N+2}\right)^{N+2} + \cdots$$

$$\leq |a_1| + \cdots + |a_{N-1}| + |a_N| + \left((l+\varepsilon)^{N+1} + (l+\varepsilon)^{N+2} + \cdots\right)$$

$$= |a_1| + \cdots + |a_{N-1}| + |a_N| + (l+\varepsilon)^{N+1}\left(1 + (l+\varepsilon) + (l+\varepsilon)^2 + \cdots\right)$$

$$= |a_1| + \cdots + |a_N| + (l+\varepsilon)^{N+1} \frac{1}{1-(l+\varepsilon)}.$$