620-295 Real Analysis with applications Lect. 36 ①

29.10.2009

Example Evaluate $\lim_{x\to 2} \dfrac{2x^2+3x-8}{x^3-2x^2+x-12}$

and justify the result using the definition of the limit.

$$\lim_{x\to 2} \frac{2x^2+3x-8}{x^3-2x^2+x-12} = \frac{2\cdot 2^2+3\cdot 2-8}{2^3-2\cdot 2^2+2-12} = \frac{6}{-10} = -\frac{3}{5}.$$

Proof

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists a $\delta \in \mathbb{R}_{>0}$ such that if $|x-2|<\delta$ ~~such that~~ then

$$\left|\frac{2x^2+3x-8}{x^3-2x^2+x-12} + \frac{3}{5}\right| < \varepsilon.$$

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists a $\delta \in \mathbb{R}_{>0}$ such that if $|x-2|<\delta$ then $\left|\dfrac{2x^2+3x-8}{x^3-2x^2+x-12} + \dfrac{3}{5}\right| < \varepsilon.$

Let $\delta = \min(\frac{1}{2}, \varepsilon/3)$.

To show: If $|x-2|<\delta$ then $\left|\dfrac{2x^2+3x-8}{x^3-2x^2+x-12} + \dfrac{3}{5}\right| < \varepsilon$

Assume $|x-2|<\delta$.

To show: $\left|\dfrac{2x^2+3x-8}{x^3-2x^2+x-12} + \dfrac{3}{5}\right| < \varepsilon.$

$$\left|\frac{2x^2+3x-8}{x^3-2x^2+x-12}+\frac{3}{5}\right| = \left|\frac{2((x-2)+2)^2+3((x-2)+2)-8}{((x-2)+2)^3-2((x-2)+2)^2+(x-2)+2-12}+\frac{3}{5}\right|$$

$$=\left|\frac{2((x-2)^2+4(x-2)+4)+3(x-2)+6-8}{(x-2)^3+6(x-2)^2+12(x-2)+8-2(x-2)^2-8(x-2)-8+(x-2)-10}+\frac{3}{5}\right|$$

$$=\left|\frac{2(x-2)^2+8(x-2)+8+3(x-2)-2}{(x-2)^3+4(x-2)^2+4(x-2)+(x-2)-10}+\frac{3}{5}\right|$$

$$=\left|\frac{2(x-2)^2+11(x-2)+6}{(x-2)^3+4(x-2)^2+5(x-2)-10}+\frac{3}{5}\right|$$

$$=\left|\frac{10(x-2)^2+55(x-2)+\cancel{30}+3(x-2)^3+12(x-2)^2+15(x-2)-\cancel{30}}{5((x-2)^3+4(x-2)^2+5(x-2)-10)}\right|$$

$$=\frac{|x-2|}{5}\left|\frac{3(x-2)^2+22(x-2)+70}{(x-2)^3+4(x-2)^2+5(x-2)-10}\right|$$

$$<\frac{\delta}{5}\cdot\cancel{3+22+70}\left|\frac{3(\frac{1}{2})^2+22\frac{1}{2}+70}{(\frac{1}{2})^3+4(\frac{1}{2})^2+5\frac{1}{2}-10}\right|$$

$$<\frac{\delta}{5}\left|\frac{82}{\frac{7}{2}+\frac{1}{8}-10}\right|<\frac{\delta}{5}\,\frac{82}{6}<\delta\,\frac{90}{30}=3\delta<\varepsilon.$$

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Example For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2} \quad \text{converge?}$$

Try a ratio test:

$$\lim_{n\to\infty}\left(\left|\frac{(2x-5)^n}{n^2}\right|\right)^{1/n} = \lim_{n\to\infty} \frac{|2x-5|}{(n^{1/n})^2} = |2x-5|,$$

since $\lim_{n\to\infty} n^{1/n} = 1$.

So the series converges if $|2x-5| < 1$.

So the series converges if $|x - \frac{5}{2}| < \frac{1}{2}$.

So the series converges if $-2 < x < 3$

The radius of convergence is $5/2$

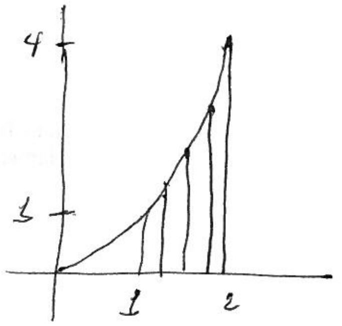
~~The~~
If $x = 3$ then the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges.

If $x = -2$ then the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges

So the series converges if $-2 \le x \le 3$.

So the interval of convergence is $[-2, 3]$.

Example Estimate $\int_1^2 x^2\,dx$ with a trapezoidal approximation with 4 slices.



$$\int_1^2 x^2\,dx \simeq \frac{\Delta x}{2}\left(f(1)+2f\left(\tfrac{5}{4}\right)+2f\left(\tfrac{6}{4}\right)+2f\left(\tfrac{7}{4}\right)+f(2)\right)$$

with $\frac{\Delta x}{2} = \frac{1}{2}\left(\frac{2-1}{4}\right) = \frac{1}{8}$ and $f(x)=x^2$.

So

$$\int_1^2 x^2\,dx \simeq \frac{1}{8}\left(1^2+2\cdot\frac{5^2}{16}+2\,\frac{6^2}{16}+2\cdot\frac{7^2}{16}+2^2\right)$$

$$= \frac{1}{8}\left(3+\frac{25+36+49}{8}\right) = \frac{1}{8}\left(\frac{49+85}{8}\right) = \frac{134}{64} = \frac{67}{32}.$$

Then

$$|\text{Error}| \leq \frac{(2-1)^3}{12\cdot 4^2}M,$$ where M is an upper bound for $f''(x)$ with $x\in(1,2)$.

Since $f''(x)=2$, $M=2$.

So $|\text{Error}| \leq \frac{1}{12\cdot 4^2}\cdot 2 = \frac{1}{6\cdot 4^2} = \frac{1}{96}$.