620-295 Real Analysis with Applications

Assignment 5: Due 5pm on 16 October

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Due 5pm on 16 October in the appropriate assignment box on the ground floor of Richard Berry.

1. (a) What is the Fundamental Theorem of Calculus?

(b) Let f(x) be a function which is continuous and let A(x) be the area under f(x) from a to x. Compute the derivative of A(x) by using limits.

(c) Why is the Fundamental Theorem of Calculus true? Explain carefully and thoroughly.

(d) Give an example which illustrates the Fundamental Theorem of Calculus. In order to do this, compute an area by summing up the areas of tiny boxes and then show that applying the Fundamental Theorem of Calculus gives the same result.

2. Using appropriate tests decided whether the following series converge absolutely, converge conditionally or diverge, giving a brief explanation:

(i)
$$\sum_{n=1}^{\infty} \frac{3^n}{n!2^n}$$

(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{\sqrt{n}}$

(iii)
$$\sum_{n=1}^{\infty} \frac{n+4}{2-3n\sqrt{n}}$$

- 3. Find the radius of convergence and the interval of convergence of the following power series: $\sum_{n=1}^{\infty} \frac{(-2)^n (x-2)^n}{n3^n}.$
- 4. You are given that $\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{2} \frac{3}{2} \frac{5}{2} \cdots \left(\frac{1}{2} + n 1\right)}{n!} x^n.$
 - (i) Find a Maclaurin expansion for $\frac{1}{\sqrt{1+x^2}}$.

(ii) Using your answer to (i) find a Maclaurin expansion for $\arcsin x$. For what values of x does your series converge?

(iii) Find a degree 3 Taylor polynomial for $\cos x$ around the point $x = \frac{1}{2}$ and give Lagrange's form of the remainder.

5. Decide whether the given sequence converges, and if so, find its limit using standard limits, limit theorems, or by referring to a continuous function.

(i)
$$a_n = \exp\left(\frac{3^n - n^2}{5^n + 7n}\right)$$

(ii) $b_n = \frac{4 - n^4}{n^3 - 7n^{\frac{1}{n}}}$

(iii)
$$c_n = \frac{\log(n+2)}{\log(n+1)}$$

6. (i) Show that $a_n = \left(\frac{n+3}{n}\right)^n$ is a bounded sequence.

(ii) Is every bounded sequence convergent? If so, give a proof; if not give a counterexample.

7. A sequence
$$(a_n)$$
 is defined by $a_{n+1} = \sqrt{2 + a_n}$ and $a_1 = 3$.

- (a) Show that $2 \le a_n \le 3$.
- (b) Show that (a_n) is monotonically decreasing.
- (c) Stating appropriate results prove that (a_n) converges and find its limit.
- 8. Evaluate the following limits, if they exist:

(i)
$$\lim_{x \to 0} \frac{\cos^2 x - 1}{x^2}$$

(ii)
$$\lim_{x \to \infty} \sqrt{x^2 + 3x} - x$$

(c) Using the definition of the limit show that $\lim_{x \to 3} \frac{x^2 - 2x - 4}{3 - x^2} = \frac{1}{6}.$

9. Classify the following improper integrals and evaluate them if they converge:

(i)
$$\int_{1}^{5} \frac{4x}{\sqrt{x^2 - 1}}$$
.

(ii)
$$\int_{1}^{\infty} \frac{1}{1+x^2}.$$

(iii) Does the following integral diverge or converge? Explain why, but do not evaluate the integral. $\int_{1}^{\infty} \frac{x^2}{(x-2)(x^{11}+2)^{1/4}}.$

10. (a) Use mathematical induction to prove that for any positive integer n, $\sum_{j=1}^{n} j(3j+1) = n$

 $(n+1)^2$.

- (b) Give the definition of a Cauchy sequence.
- (c) Prove that every convergent sequence is Cauchy.