620-295 Real Analysis with Applications

Assignment 6: Due 5pm on 30 October

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Due 5pm on 30 October in the appropriate assignment box on the ground floor of Richard Berry.

1. Determine the area of a parabola topped slice with left edge at $x = l$, right edge at $x = l + 2\Delta x$, middle at $x = l + \Delta x$, left height $f(l)$, middle height $f(l + \Delta x)$, and right height $f(l + 2\Delta x)$.

2. Assume that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Show that $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

3. Assume that $\lim_{x \to a} f(x)$ exists. Show that $\lim_{x \to a} \exp(f(x)) = \exp\left(\lim_{x \to a} f(x)\right)$.

4. Assume that $f(x) = c_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$. Show that $a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx$, $a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx$ and $b_{\mathrm{k}} = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx$.

5. Write a quadratic approximation for $f(x) = x^{1/3}$ near 8 and approximate $9^{1/3}$. Estimate the error and find the smallest interval that you can be sure contains the value.

6. Define the following and give an example of each:
   (a) converges pointwise
   (b) converges uniformly
   (b) Taylor series
   (b) Maclaurin series
   (b) Lagrange's remainder
   (b) Riemann's integral
   (b) Trapezoidal integral
   (b) Simpson's integral

7. Carefully state and prove the mean value theorem.

8. (a) Define topological space.
   (b) Define closure of a set.
   (b) Define close point.

(d) Let X be a topological space and let E be a subset of X. Show that the closure of E is equal to the set of close points to E.

9. Assume that $f : \left[a, b\right] \to \mathbb{R}$ and $g : \left[a, b\right] \to \mathbb{R}$ are functions, if $x \in [a, b]$ then $f(x) \leq g(x)$, and $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist. Show that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

10. Assume that $f : \left[a, b\right] \to \mathbb{R}$ is continuous. Show that there exists $c \in [a, b]$ such that if $x \in [a, b]$ then $f(x) \leq f(c)$ (i.e. f has a maximum at c).