Problem Set -- Numbers and Ordered fields
620-295 Semester I 2010

Arun Ram
Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3010 Australia
aram@unimelb.edu.au
and

Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

Last updates: 2 May 2010

1. Integers $\mathbb{Z}$

(1) Determine the solutions to the equation $m+1=2$ if $m \in \mathbb{Z}_{>0}$, if $m \in \mathbb{Z}_{\geq 0}$, if $m \in \mathbb{Z}$, and if $m \in \mathbb{Q}$.

(2) Determine the solutions to the equation $m+2=1$. if $m \in \mathbb{Z}_{>0}$, if $m \in \mathbb{Z}_{\geq 0}$, if $m \in \mathbb{Z}$, and if $m \in \mathbb{Q}$.

(3) Determine the solutions to the equation $2m=4$ if $m \in \mathbb{Z}_{>0}$, if $m \in \mathbb{Z}_{\geq 0}$, if $m \in \mathbb{Z}$, and if $m \in \mathbb{Q}$.

(4) Determine the solutions to the equation $2m=3$ if $m \in \mathbb{Z}_{>0}$, if $m \in \mathbb{Z}_{\geq 0}$, if $m \in \mathbb{Z}$, and if $m \in \mathbb{Q}$.

(5) Determine the solutions to the equation $0 \cdot m = 42$ if $m \in \mathbb{Z}_{>0}$, if $m \in \mathbb{Z}_{\geq 0}$, if $m \in \mathbb{Z}$, and if $m \in \mathbb{Q}$.

(6) Determine the solutions to the equation $0 \cdot m = 0$ if $m \in \mathbb{Z}_{>0}$, if $m \in \mathbb{Z}_{\geq 0}$, if $m \in \mathbb{Z}$, and if $m \in \mathbb{Q}$.

(7) Define *even* and prove that if the square of an integer is even then the integer itself is even.

(8) Prove that if $m \in \mathbb{Z}_{>0}$ and m is divisible by 12 then its square is also divisible by 12.

(9) Prove that if $m \in \mathbb{Z}_{>0}$ and m^2 is divisible by 12 then m also divisible by 12.

(10) Describe the subset S of $\mathbb{Z}_{>0}$ such that if $a \in S$ and $n \in \mathbb{Z}_{>0}$ then n is divisible by a if and only if n^2 is divisible by a.

(11) Prove that, for any 1993 integers, there is a subset whose sum is divisible by 1993.

(12) Prove that the square of an even integer is even.

(13) Prove that the product of two odd integers is odd.

(14) Prove that the sum of two odd integers is even.

(15) Prove that the cube of an odd integer is odd.

(16) Prove that if k is an odd integer then $k^2 - 1$ is divisible by 4.

(17) Rewrite $2 + 4 + 6 + \cdots + 2n$ in summation notation.

(18) Rewrite $1 + 4 + 7 + \cdots + (3n - 2)$ in summation notation.

(19) Rewrite $2 + 7 + 12 + \cdots + (5n - 3)$ in summation notation.

(20) Rewrite $1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + n2^{n-1}$ in summation notation.

(21) Rewrite $1^2 + 2^2 + 3^2 + \cdots + n^2$ in summation notation.

(22) Rewrite $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$ in summation notation.

(23) Rewrite $3 + 3^2 + 3^3 + \cdots + 3^n$ in summation notation.

(24) Rewrite $(1 + 2^5 + \cdots + n^5) + (1 + 2^7 + \cdots + n^7)$ in summation notation.

(25) Rewrite $1 + r + r^2 + \cdots + r^n$ in summation notation.

(26) Compute $\sum_{k=1}^{n} 1$.

(27) Compute $\sum_{k=1}^{n} k$.

(28) Let $a, r \in \mathbb{R}$. Compute $\sum_{k=1}^{n} ar^k$.

(29) Given 5 points on a square of side length 1, show that there are two points of the five for which the distance apart is no more than $\frac{\sqrt{2}}{2}$.

(30) Suppose that the points of the plane are each colored eqither red, yellow or blue. Prove that there are two points at distance one apart which have the same color.

(31) Assume that the area of a square of side length a is a^2. State and prove the Pythagorean theorem.

2. Rational numbers $\mathbb{Q}$

(1) Give an example of $s \in \mathbb{Q}$ which has more than one representation as a fraction.

(2) Show that $\sqrt{2} \notin \mathbb{Q}$.

(3) Show that $\sqrt{3} \notin \mathbb{Q}$.

(4) Show that $\sqrt{15} \notin \mathbb{Q}$.

(5) Show that $2^{1/3} \notin \mathbb{Q}$.

(6) Show that $11^{1/4} \notin \mathbb{Q}$.

(7) Show that $16^{1/5} \notin \mathbb{Q}$.

(8) Show that $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

(9) Show that the number $e^1 = 2.71828...$ is irrational by using the series expansion

$$e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \quad .$$

If $e^1 = p/q$ where p and q are positive integers, consider $q!e^1$ to get a contradiction.

(10) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Define carefully what $\frac{a}{b} = \frac{c}{d}$ means and prove that if $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$ then $\frac{a}{b} = \frac{e}{f}$.

(11) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Define carefully $\frac{a}{b} + \frac{c}{d}$ and $\frac{a}{b} \cdot \frac{c}{d}$.

(12) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Prove carefully that if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$.

(13) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Prove carefully that if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}$.

(14) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Prove carefully $\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f}$.

(15) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Prove carefully that if $\frac{a}{b} + \frac{c}{d} = \frac{c}{d}$ then $\frac{a}{b} = \frac{0}{1}$.

(16) Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Prove carefully that if $\frac{a}{b} + \frac{c}{d} = \frac{0}{7}$ then $\frac{c}{d} = \frac{-a}{b}$.

(17) Compute $\left(27^{\frac{1}{3}}\right)^4$ and $27^{\left(4+\frac{1}{3}\right)}$ and graph the result.

(18) Compute $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$.

(19) Compute $1 \cdot 2$, $1 \cdot 2 \cdot 3$, $1 \cdot 2 \cdot 3 \cdot 4$, $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ and $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$.

(20) Compute $\frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \cdots$.

3. The real numbers $\mathbb{R}$

(1) Give an example of $s \in \mathbb{R}$ which has more than one decimal expansion.

(2) Show that .9999... = 1.00000... .

(3) Show that the sum of two irrational numbers need not be irrational.

(4) Show that the product of two irrational numbers need not be irrational.

(5) Compute the decimal expansion of $\frac{3651}{342}$.

(6) Let $x, y \in \mathbb{R}$ be given by

$x = .98765432109876543210987...$ and $y = 1.01001000100001....$

Compute the first 10 decimal places of xy.

(7) Let $f : \mathbb{Q} \to \mathbb{R}$ be a function such that $f(1/1) = 1.000...$, $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$.

(a) Show that $f(1/8) = 0.125000....$

(b) Show that f is not surjective.

(8) Express 0.1111... as a rational number.

(9) Express 2.6666... as a rational number.

(10) Express 0.9999... as a rational number.

(11) Express 0.349999... as a rational number.

(12) Express 0.37373737... as a rational number.

(13) Express 0.00101010101... as a rational number.

(14) Give an example of a decimal expansion that cannot be expressed as a rational number.

(15) Show that the decimal expansion of a rational number is eventually repeating.

(16) Show that any decimal expansion which is eventually repeating represents a rational number.

(17) State and prove the Pythagorean Theorem.

(18) Compute the decimal expansion of $\sqrt{2}$ to 10 digits.

(19) Compute the decimal expansion of π to 10 digits.

(20) Compute the decimal expansion of $2\sqrt{2}$ to 10 digits.

(21) Compute the decimal expansion of π^2 to 10 digits.

(22) Compute the decimal expansion of $-\sqrt{2}$ to 10 digits.

(23) Compute the decimal expansion of $1/\sqrt{2}$ to 10 digits.

4. Complex Numbers $\mathbb{C}$

(1) Define the following sets and give examples of elements of each:

(a) the set of positive integers,
(b) the set of nonnegative integers,
(c) the set of integers,
(d) the set of rational numbers,
(e) the set of real numbers,
(f) the set of complex numbers,
(g) the set of algebraic numbers.

(2) Find a complex number z such that $z + w = w$ for all complex numbers w.

(3) Find a complex number z such that $zw = w$ for all complex numbers w.

(4) Graph $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}$, $\mathbb{R}$, and $\overline{\mathbb{Q}}$, as subsets of $\mathbb{C}$.

(5) State the fundamental theorem of algebra.

(6) Compute $(3 - 7i) + (2 + 5i)$ and graph the result.

(7) Compute $(-12+3i)-(7-5i)$ and graph the result.

(8) Compute $(4+8i)(2-3i)$ and graph the result.

(9) Compute $\dfrac{-15+i}{4+2i}$ and graph the result.

(10) Compute $(3-2i)^3$ and graph the result.

(11) Compute $\sqrt{2i}$ and graph the result.

(12) Compute $\dfrac{1}{a+bi}$ and graph the result, where $a,\ b \in \mathbb{R}$.

(13) Compute $(3-5i)+(7+2i)$ and graph the result.

(14) Compute $(5-2i)-(3-6i)$ and graph the result.

(15) Compute $(2-4i)(3+2i)$ and graph the result.

(16) Compute $\dfrac{6-i}{4+2i}$ and graph the result.

(17) Compute $1^{\frac{1}{4}}$ and graph the result.

(18) Compute $16^{\frac{1}{4}}$ and graph the result.

(19) Compute and graph $\left(27^{1/3}\right)^4$.

(20) Compute and graph $27^{(4+1/3)}$.

(21) Compute and graph $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^3$.

(22) Compute and graph $(1+i)^n+(1-i)^n$, for $n \in \mathbb{Z}_{\geq 0}$.

(23) Let $z = x + yi$ with $x,\ y \in \mathbb{R}$. Show that

$$z^{-1} = \frac{1}{|z|^2}(x - yi).$$

(24) Let $z = x + iy$ with $x,\ y \in \mathbb{R}$. Compute and graph $\dfrac{1}{z}$.

(25) Let $z = x + iy$ with $x,\ y \in \mathbb{R}$. Compute and graph z^4.

(26) Let $z = x + iy$ with $x,\ y \in \mathbb{R}$. Compute and graph $\left|\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right|$.

(27) Show that the conjugate of $\frac{z}{z^2+1}$ is equal to $\frac{\bar{z}}{\bar{z}^2+1}$.

5. Fields and Ordered fields

(1) Define (a) field and (b) ordered field.

(2) Let $\mathbb{F}$ be a field. Prove that if $x \in \mathbb{F}$ then $0 \cdot x = x$.

(3) Let $\mathbb{F}$ be a field. Prove that if $x \in \mathbb{F}$ then $-(-x) = x$.

(4) Let $\mathbb{F}$ be a field. Prove that if $x \in \mathbb{F}$ and $x \neq 0$ then $\left(x^{-1}\right)^{-1} = x$.

(5) Let $\mathbb{F}$ be a field. Prove that if $x \in \mathbb{F}$ then $x \cdot (-1) = -x$.

(6) Let $\mathbb{F}$ be a field. Prove that if $x, y \in \mathbb{F}$ then $(-x)y = -xy$.

(7) Let $\mathbb{F}$ be a field. Prove that if $x, y \in \mathbb{F}$ then $(-x)(-y) = xy$.

(8) Let $\mathbb{F}$ be an ordered field. Prove that if $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.

(9) Let $\mathbb{F}$ be an ordered field. Prove that if $a \in \mathbb{F}$ and $a \neq 0$ then $a^2 > 0$.

(10) Let $\mathbb{F}$ be an ordered field. Prove that if $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.

(11) Let $\mathbb{F}$ be an ordered field. Prove that if $a, b \in \mathbb{F}$ and $a > 0$ and $b > 0$ then $ab > 0$.

(12) Let $\mathbb{F}$ be an ordered field. Prove that if $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$ then $a + b \geq 0$.

(13) Let $\mathbb{F}$ be an ordered field. Prove that if $a, b \in \mathbb{F}$ and $0 < a < b$ and then $b^{-1} < a^{-1}$.

(14) Let $\mathbb{F}$ be an ordered field. Prove that if $a, b \in \mathbb{F}$ and $a \leq b$ then $a^2 \leq b^2$.

(15) Let $\mathbb{F}$ be an ordered field. Prove that if $a, b \in \mathbb{F}$ and $a^2 \leq b^2$ then $a \leq b$.

(16) Let $\mathbb{F}$ be an ordered field. Prove that $1 > 0$.

6. References

[Ca] S. Carnie, *620-143 Applied Mathematics, Course materials*, 2006 and 2007.

[Ho] C. Hodgson, *620-194 Mathematics B and 620-211 Mathematics 2 Notes*, Semester 1, 2005.

[Hu] B.D. Hughes, *620-158 Accelerated Mathematics 2 Lectures*, 2009.

[Wi] P. Wightwick, *UMEP notes*, 2010.