# Week 8 Problem Sheet
# Group Theory and Linear algebra
# Semester II 2011

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Last updates: 1 September 2011



## 1. Week 8: Vocabulary

(1) Let $G$ be a group and let $H$ be a subgroup. Define a left coset of $H$, a right coset of $H$ and the index of $H$ in $G$ and give some illustrative examples.

(2) Let $G$ be a group and let $H$ be a subgroup. Define $G/H$ and give some illustrative examples.

(3) Let $G$ be a group. Define normal subgroup of $G$ and give some illustrative examples.

(4) Let $G$ be a group and let $H$ be a normal subgroup. Define the quotient group $G/H$ and give some illustrative examples.

## 2. Week 8: Results

(1) Let $G$ be a group and let $H$ be a subgroup of $G$. Let $a, b \in G$. Show that $Ha = Hb$ if and only if $ab^{-1} \in H$.

(2) Let $G$ be a group and let $H$ be a subgroup of $G$. Show that each element of $G$ lies in exactly one coset of $G$.

(3) Let $G$ be a group and let $H$ be a subgroup of $G$. Let $a, b \in G$. Show that the function $f: Ha \to Hb$ given by $f(ha) = hb$ is a bijection.

(4) Let $G$ be a group and let $H$ be a subgroup of $G$. Show that $G/H$ is a partition of $G$.

(5) Let $G$ be a group and let $H$ be a subgroup of $G$. Let $g \in G$. Show that $gH$ and $H$ have the same number of elements.

(6) Let $G$ be a group of finite order and let $H$ be a subgroup of $G$. Show that $\mathrm{Card}(H)$ divides $\mathrm{Card}(G)$.

(7) Let $G$ be a group of finite order and let $g \in G$. Show that the order of $g$ divides the order of $G$.

(8) Let $G$ be a finite group and let $n = \mathrm{Card}(G)$. Show that if $g \in G$ then $g^n = 1$.

(9) Let $p$ be a prime positive integer. Show that if $a$ is an integer which is not a multiple of $p$ then $a^{p-1} = 1 \bmod p$.

(10) Let $p$ be a prime positive integer. Let $G$ be a group of order $p$. Show that $G$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(11) Let $G$ be a group and let $H$ be a subgroup of $G$. Show that $H$ is a normal subgroup of $G$ if and only if $H$ satisfies

$$\text{if} \quad g \in G \qquad \text{then} \quad Hg = gH\,.$$

(12) Let $G$ be a group and let $H$ be a subgroup of $G$. Show that $H$ is a normal subgroup of $G$ if and only if $H$ satisfies

$$\text{if} \quad g \in G \qquad \text{then} \quad gHg^{-1} = H\,.$$

(13) Let $G$ be a group and let $H$ be a normal subgroup of $G$. Show that if $a, b \in G$ then $HaHb = Hab$.

(14) Let $G$ be a group and let $H$ be a normal subgroup of $G$. Show that $G/H$ with operation given by $(g_1H)(g_2H) = g_1g_2H$ is a group.

(15) Let $f: G \to H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of $G$.

(16) Let $f: G \to H$ be a group homomorphism. Show that $\mathrm{im}\, f$ is a subgroup of $H$.

(17) Let $f: G \to H$ be a group homomorphism. Show that $f$ is injective if and only if $\ker f = \{1\}$.

(18) Let $G$ be a group and let $H$ be a normal subgroup of $G$. Let $f: G \to G/H$ be given by $f(g) = gH$. Show that

- (a) $f$ is a group homomorphism,
- (b) $\ker f = H$,
- (c) $\mathrm{im}\, f = G/H$.

(19) Let $f: G \to H$ be a group homomorphism. Show that $G/\ker f \cong \mathrm{im}\, f$.

# 3. Week 8: Examples and computations

(1) Let
$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
Show that $A$ has order 3, that $B$ has order 4 and that $AB$ has infinite order.

(2) Assume that $G$ is a group such that
$$\text{if } g, h \in G \qquad \text{then} \quad (gh)^2 = g^2h^2 .$$
Show that $G$ is commutative.

(3) Decide whether the positive integers is a subgroup of the integers with operation addition.

(4) Decide whether the set of permutations which fix 1 is a subgroup of $S_n$.

(5) List all subgroups of $\mathbb{Z}/12\mathbb{Z}$.

(6) Let $G$ be a group, let $H$ be a subgroup and let $g \in G$. Show that $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$ is a subgroup of $G$.

(7) Let $G$ be a group and let $g \in G$. Let $f: G \to G$ be given by $f(h) = ghg^{-1}$. Show that $f$ is an isomorphism.

(8) Show that $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to $U_1(\mathbb{C})$.

(9) Show that $(\mathbb{R}, +)$ and $(\mathbb{R}^\times, \times)$ are not isomorphic.

(10) Show that $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic.

(11) Show that $(\mathbb{Z}, +)$ and $(\mathbb{Q}_{>0}, \times)$ are not isomorphic.

(12) Show that $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

(13) Find the orders of elements $1, -1, 2$ and $i$ in the group $\mathbb{C}^\times = \mathbb{C} - \{0\}$ with operation multiplication.

(14) Find the orders of elements in $\mathbb{Z}/6\mathbb{Z}$.

(15) Find the subgroups of $\mathbb{Z}/6\mathbb{Z}$.

(16) Write the element $(345)$ in $S_5$ in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(17) Write the element $(13425)$ in $S_5$ in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(18) Write the element (13)(24) in $S_5$ in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(19) Write the element (12)(345) in $S_5$ in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(20) Let $n$ be a positive integer. Determine if the group of complex $n$th roots of unity $\{z \in \mathbb{C} \mid z^n = 1\}$ (with operation multiplication) is a cyclic group.

(21) Determine if the rational numbers $\mathbb{Q}$ with operation addition is a cyclic group.

(22) Find the order of the element (1,2) in the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

(23) Show that the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and the group $\mathbb{Z}/12\mathbb{Z}$ are not isomorphic.

(24) Show that the group $\mathbb{Z} \times \mathbb{Z}$ and the group $\mathbb{Q}$ with operation addition are not isomorphic.

(25) Let $G$ be a group and let $a, b \in G$. Assume that $ab = ba$.

(a) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $ab^n = b^n a$,
(b) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $a^n b^n = b^n a^n$,
(c) Show that the order of $ab$ divides the least common multiple of the order of $a$ and the order of $b$.
(d) Show that if $a = (12)$ and $b = (13)$ then the order of $ab$ does not divide the least common multiple of the order of $a$ and the order of $b$.

(26) Show that the order of $GL_2(\mathbb{Z}/2\mathbb{Z})$ is 6.

(27) Let $p$ be a prime positive integer. Find the order of the group $GL_2(\mathbb{Z}/p\mathbb{Z})$.

(28) Let $n \in \mathbb{Z}_{>0}$ and let $p$ be a prime positive integer. Find the order of the group $GL_n(\mathbb{Z}/p\mathbb{Z})$.

(29) Show that the group $\mathbb{Z}[x]$ of polynomials with integer coefficients with operation addition is isomorphic to the group $\mathbb{Q}_{>0}$ with operation multiplication.

(30) Let $G$ be a group with less than 100 elements which has subgroups of orders 10 and 25. Find the order of $G$.

(31) Let $G$ be a group and let $H$ and $K$ be subgroups of $G$. Show that $|H \cap K|$ is a common divisor of $|H|$ and $|K|$.

(32) Let $G$ be a group and let $H$ and $K$ be subgroups of $G$. Assume that $|H| = 7$ and $|K| = 29$. Show that $H \cap K = \{1\}$.

(33) Let $H$ be the subgroup of $G = \mathbb{Z}/6\mathbb{Z}$ generated by 3. Compute the right cosets of $H$ in $G$ and the index $|G:H|$.

(34) Let $H$ be the subgroup of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ generated by $(1, 0)$. Find the order of each element in $G/H$ and identify the group $G/H$.

(35) Let $H$ be the subgroup of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ generated by $(0, 2)$. Find the order of each element in $G/H$ and identify the group $G/H$.

(36) Let $n \in \mathbb{Z}_{\geq 2}$ and define $f: \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ by $f(A) = A^t$. Determine whether $f$ is a group homomorphism.

(37) Let $n \in \mathbb{Z}_{\geq 2}$ and define $f: \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ by $f(A) = \left(A^{-1}\right)^t$. Determine whether $f$ is a group homomorphism.

(38) Let $n \in \mathbb{Z}_{\geq 2}$ and define $f: \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ by $f(A) = A^2$. Determine whether $f$ is a group homomorphism.

(39) Let $B$ be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ of upper triangular matrices and let $T$ be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ of diagonal matrices. Let $f: B \to T$ be given by

$$f\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Show that $f$ is a group homomorphism. Find $N = \ker f$ and identify the quotient $B/N$.

(40) Assume $G$ is a cyclic group and let $N$ be a subgroup of $G$. Show that $N$ is a normal subgroup of $G$ and that $G/N$ is a cyclic group.

(41) Simplify $3^{52} \bmod 53$.

(42) Suppose that $2^{147052} = 76511 \bmod 147053$. What can you conclude about 147053?

(43) Show that if $f: G \to H$ is a group homomorphism and $a_1, a_2, \ldots, a_n \in G$ then $f(a_1 a_2 \cdots a_n) = f(a_1) f(a_2) \cdots f(a_n)$.

(44) Describe all group homomorphisms $f: \mathbb{Z} \to \mathbb{Z}$.

(45) Show that $\mathrm{SO}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{O}_n(\mathbb{R})$ by finding a homomorphism $f: \mathrm{O}_n(\mathbb{R}) \to \{\pm 1\}$ with kernel $\mathrm{SO}_n(\mathbb{R})$. Identify the quotient $\mathrm{O}_n(\mathbb{R}) / \mathrm{SO}_n(\mathbb{R})$.

(46) Show that $\mathrm{SU}_n(\mathbb{C})$ is a normal subgroup of $\mathrm{U}_n(\mathbb{C})$ by finding a homomorphism $f: \mathrm{U}_n(\mathbb{C}) \to \mathrm{U}_1(\mathbb{C})$ with kernel $\mathrm{SU}_n(\mathbb{C})$. Identify the quotient $\mathrm{U}_n(\mathbb{C}) / \mathrm{SU}_n(\mathbb{C})$.

(47) Let $G$ be a group and let $H$ be a subgroup of $G$. Let $f: G/H \to H \backslash G$ be given by $f(aH) = Ha^{-1}$. Show that $f$ is a function and that $f$ is a bijection.

(48) Let $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(49) Let $G = S_3$ and let $H$ be the subgroup generated by $(123)$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(50) Let $G = S_3$ and let $H$ be the subgroup generated by $(12)$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(51) Let $G = \mathrm{GL}_2(\mathbb{R})$ and let $H = \mathrm{SL}_2(\mathbb{R})$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(52) Let $G$ be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ given by

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R}, x > 0 \right\}$$

Let $H$ be the subgroup of $G$ given by

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, z \in \mathbb{R}, z > 0 \right\}.$$

Each element of $G$ can be identified with a point $(x, y)$ of $\mathbb{R}^2$. Use this to describe the right cosets of $H$ in $G$ geometrically. Do the same for the left cosets of $H$ in $G$.

(53) Consider the set $AX = B$ of linear equations where $X$ and $B$ are column vectors, $X$ is the matrix of unknowns, and $A$ the matrix of coefficients. Let $W$ be the subspace of $\mathbb{R}^n$ which is the set of solutions of the homogeneous equations $AX = 0$. Show that the set of solutions of $AX = B$ is either empty or is a coset of $W$ in the group $\mathbb{R}^n$ (with operation addition).

(54) Let $H$ be a subgroup of index 2 in a group $G$. Show that if $a, b \in G$ and $a \notin H$ and $b \notin H$ then $ab \in H$.

(55) Let $G$ be a group. Let $H$ be a subgroup of $G$ such that if $a, b \in G$ and $a \notin H$ and $b \notin H$ then $ab \in H$. Show that $H$ has index 2 in $G$.

(56) Let $G$ be a group of order $841 = (29)^2$. Assume that $G$ is not cyclic. Show that if $g \in G$ then $g^{29} = 1$.

(57) Show that the subgroup $\{(1), (123), (132)\}$ of $S_3$ is a normal subgroup.

(58) Show that the subgroup $\{(1), (12)\}$ of $S_3$ is not a normal subgroup.

(59) Show that $\mathrm{SL}_n(\mathbb{C})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{C})$.

(60) Let $G$ be a group. Show that $\{1\}$ and $G$ are normal subgroups of $G$.

(61) Show that every subgroup of an abelian group is normal.

(62) Write down the cosets in $\mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C})$ then show that

$$\mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C}) \simeq \mathrm{GL}_1(\mathbb{C}).$$

(63) Show that the function $\det\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{C})$ given by taking the determinant of a matrix is a homomorphism.

(64) Show that the function $f\colon \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{R})$ given by $f(z) = |z|$ is a homomorphism.

(65) Show that the determinant function $\det\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{C})$ is surjective and has kernel $\mathrm{SL}_n(\mathbb{C})$.

(66) Show that the homomorphism $f\colon \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{R})$ given by $f(z) = |z|$ has image $\mathbb{R}_{>0}$ and kernel $\mathrm{U}_1(\mathbb{C})$ (the group of $1 \times 1$ unitary matrices. Conclude that

$$\mathrm{GL}_1(\mathbb{C})/\mathrm{U}_1(\mathbb{C}) \simeq \mathbb{R}_{>0}.$$

(67) Show that the homomorphism

$$\begin{array}{rccc} f\colon & \mathbb{R} & \to & \mathrm{SO}_2(\mathbb{R}) \\ & \theta & \to & \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{array}$$

is surjective with kernel $2\pi\,\mathbb{Z}$. Conclude that

$$\mathbb{R}/(2\pi\,\mathbb{Z}) \simeq \mathrm{SO}_2(\mathbb{R}).$$

(68) Show that the set of matrices $H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, ad \neq 0 \right\}$ is a subgroup of GL2($\mathbb{R}$) and that the set of matrices $K = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}$ is a normal subgroup of $H$.

(69) Let $G$ be a group and let $H$ be a subgroup of $G$. Show that $HH = H$.

(70) Let $G$ be a group and let $K$ and $L$ be normal subgroups of $G$. Show that $K \cap L$ is a normal subgroup of $G$.

(71) Let $G$ be a group and let $n$ be a positive integer. Assume that $H$ is the only subgroup of $G$ of order $n$. Show that $H$ is a normal subgroup of $G$.

(72) Let $G$ be an abelian group and let $N$ be a normal subgroup of $G$. Show that $G/N$ is abelian.

(73) Let $G$ be a cyclic group and let $N$ be a normal subgroup of $G$. Show that $G/N$ is cyclic.

(74) Find surjective homomorphisms from $\mathbb{Z}/8\mathbb{Z}$ to $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and $\{1\}$ (the group with one element).

(75) Let $\mathbb{R}$ denote the group of real numbers with the operation of addition and let $\mathbb{Q}$ and $\mathbb{Z}$ be the subgroups of rational numbers and integers, respectively. Show that it is possible to regard $\mathbb{Q}/\mathbb{Z}$ as a subgroup of $\mathbb{R}/\mathbb{Z}$ and show that this subgroup consists exactly of the elements of finite order in $\mathbb{R}/\mathbb{Z}$.

# 4. References

[GH] J.R.J. Groves and C.D. Hodgson, *Notes for 620-297: Group Theory and Linear Algebra*, 2009.

[Ra] A. Ram, *Notes in abstract algebra*, University of Wisconsin, Madison 1994.