Week 11 Problem Sheet
Group Theory and Linear algebra
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1. Week 11: Questions from past exams

(1) Give an example of a 3×3 Hermitian matrix over the complex numbers in which not all entries are real.

(2) Let f and g be linear transformations on a finite dimensional inner product space. If f^* and g^* denote the adjoints of f and g, respectively, show that

$$(fg)^* = g^* f^*$$

where $(fg)^*$ denotes the adjoint of fg.

(3) Give the product of the following permutations in S_8 and the order of the result: (12)(3456) and (16483725).

(4) Give an example, with detailed explanation, of an infinite noncommutative group.

(5) Let $\mathbb{Z}/7\mathbb{Z} - \{0\}$ denote the group of nonzero elements of the integers modulo 7 together with the operation of multiplication. Show that this group is cyclic.

(6) Give an example of a group G and a subgroup H of G which is *not* normal in G.

(7) In the group D_4 of all symmetries of a square, let g be an element which represents a reflection in a diagonal of the square. Describe the conjugacy class containing g.

(8) Show that, it a linear transformation f on a complex inner product space V has diagonal matrix with respect to some orthonormal basis of V then f is normal.

(9) State the Spectral Theorem for a linear transformation on a complex vector space. Use it to deduce that any normal matrix A which satisfies $A^n = 0$ for some n must satisfy $A = 0$.

(10) A regular pentagon has as its symmetry group the group D_5 with 10 elements. Suppose that the edges of the pentagon a coloured in some way in which each edge may be coloured with one or several colours. Show that the set of elements of D_5 which map the pentagon onto itself, with the same colouring, is a subgroup of D_5. Give the possible orders of such subgroups. For each such order give a coloured pentagon which has a symmetry group of that order.

(11) Let $\mathbb{R}$ denote the group of real numbers under the operation of addition. Let U denote the group of complex numbers with absolute value 1 under the operation of multiplication. Prove that the function $\mathbb{R} \to U$ given by $a \mapsto e^{2\pi i a}$ for all $a \in \mathbb{R}$ is a homomorphism. Deduce that the quotient group $\mathbb{R} / \mathbb{Z}$ is isomorphic to U.

(12) Let $\mathcal{J}$ denote the group of all isometries of the plane. Let P be some fixed point in the plane. Show that any element of $\mathcal{J}$ can be uniquely expressed as a product of a translation and an isometry which fixes P.

(13)

(a) Give an example of a diagonal 3×3 matrix over the complex numbers which is unitary but not Hermitian.

(b) Give an example of a diagonal 3×3 matrix over the complex numbers which is Hermitian but not unitary.

(14) Let f be an isometric linear transformation on a finite dimensional complex inner product space V; thus f satisfies $f^* f = \mathrm{id}_V$. Explain why $\|f(v)\| = \|v\|$ for all $v \in V$.

(15) Find the order of the permutation (15749)(23)(68). What is the 12th power of this permutation?

(16) Give an example, with explanation, of a non-abelian group which contains an element of order 4.

(17) Let G denote the group of all 3×3 diagonal matrices where each diagonal entry is 1 or -1, with the operation of matrix multiplication. Decide, and explain, whether G is a cyclic group.

(18) A group is known to contain an element of order 2, an element of order 3, and an element of order 5. What is the least possible order for this group?

(19) How many conjugates does the permutation (123) have in the group S_3 of all permutations on 3 letters? Give brief reasons for your answer.

(20) Let G denote the group of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \qquad a, b, c \in \mathbb{R}, \quad a \neq 0, \quad c \neq 0,$$

with the operation of multiplicatio and let H denote the group of all matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad a, c \in \mathbb{R}, \quad a \neq 0, \quad c \neq 0,$$

also with the operation of multiplication.
Show that the function $f: G \to H$ given by

$$f\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

is a homomorphism. Find the kernel and image of this homomorphism. Hence show that H is isomorphic to a quotient group of G.

(21) Let X denote the set of n axes of symmetry of a regular n-gon. Let Dn denote the group of all symmetries of a regular n-gon; thus D_n contains $2n$ elements. Explain briefly what it means to say that D_n acts on X. If n is even, use this action to show that D_n has a normal subgroup of order 2.

(22) Consider the one dimensional pattern below

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which is assumed to be repeated indefinitely in both horizontal directions.
Find the symmetry group of this pattern. Identify the translation subgroup and the point group. Is the symmetry group abelian?

(23) Give an example of an inner product on the space $\mathscr{P}_4(\mathbb{R})$ of all polynomials with real coefficients and degree at most 4. Check that this is an inner product.

(24) If f is a linear transformation on an inner product space and if f^* denotes the adjoint of f, show that $(f^*)^* = f$.

(25) Find the order of the product of the two permutations (167253)(48) with (1645)(28).

(26) Give an example of an abelian group which is not cyclic. Always explain your answers.

(27) You are told that an icosahedron has a total of 120 symmetries. Explain how to deduce, *from this information alone*, that there is no rotational symmetry of order 9.

(28) Suppose that G is an abelian group. If N is a normal subgroup of G, expalin briefly why the quotient group G/N is also abelian.

(29) Let $\mathrm{GL}_2(\mathbb{R})$ denote the group of all invertible 2×2 matrices with real entries. Give an example of a matrix in $\mathrm{GL}_2(\mathbb{R})$ which is conjugate, but not equal to, the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(30) Let f be a normal linear transformation on an inner product space V. Let f^* denote the adjoint of f. If $v \in V$ show that

$$\langle f(v), f(v) \rangle = \langle f^*(v), f^*(v) \rangle$$

and use this to deduce that the kernel of f is equal to the kernel of f^*.

(31) Let $n = 2^k$ for some natural number k. Let U_n be the set of elements $a \in \mathbb{Z}/n\mathbb{Z}$ with a *odd*. When U_n is furnished with the operation of multiplication modulo n, then the result is a group.
What is the order of U_n? Use this to deduce that, if a is an odd integer, then $a^{2^{k-1}} - 1$ is a multiple of 2^k.

(32) Let $\mathbb{C}^*$ denote the group of non-zero complex numbers with the operation of multiplication and let $\mathbb{R}^*$ denote the group of non-zero real numbers with the operation of multiplication. Consider the function $\mathrm{abs}: \mathbb{C}^* \to \mathbb{R}^*$ given by $\mathrm{abs}(z) = |z|$. Identify the kernel and image of abs and use this to show that a quotient group of $\mathbb{C}^*$ is isomorphic to a subgroup of $\mathbb{R}^*$.

(33) Suppose that a group of order 35 acts on a set X with 18 elements. Show that some element of X must be left fixed by every permutation corresponding to an element of G. Give an example of a group of order 35 which acts on a set with 12 elements in such a way that no element is left fixed by every permutation corresponding to an element of G
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(34) Let V be the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, 1, 0)$, $(0, 1, 2)$. Find the orthogonal complement of V, using the dot product as inner product on $\mathbb{R}^3$.

(35) Give an example of a 2×2 Hermitian matrix in which not all entries are real. Is your matrix normal? Provide a thorough explanation.

(36) Does the set $P = \{x \in \mathbb{R} \mid x > 0\}$ of positive real numbers form a group using the operation of: (i) addition, (ii) multiplication? Always explain your answers.

(37) Calculate the product of the following permutations, and find the order of the result: (135)(2678) and (14)(23578).

(38) The set $H = \{1, 2, 4, 5, 7, 8\}$ of elements from $\mathbb{Z}/9\mathbb{Z}$ forms a group under multiplication modulo 9. Show that this group is cyclic and find a generator for the group.

(39) A finite group G has fewer than 100 elements and has subgroups of orders 10 and 25. What is the order of G? Always explain your answers.

(40) Let G be the group of rotational symmetries of a cube, and consider the action of G on the set X of all 6 faces of the cube. Describe the orbit and stabiliser of a face F. Use this to find the order of G. Always explain your answers.

(41) State the Spectral Theorem for a linear transformation on a complex vector space. Use it to show that every normal matrix A with complex entries has a cube root, i.e. there is a matrix B with complex entries such that $B^3 = A$.

(42) Let f be an isometry on a complex inner product space V, i.e. f is an linear transformation satisfying $f^* f = 1$.

(a) Show that each eigenvalue λ of f satisfies $|\lambda| = 1$.

(b) Show that if v_1, v_2 are eigenvectors of f corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$, then v_1, v_2 are orthogonal.

(43) Let G be a group in which every element satisfies $g^2 = 1$.

(a) Prove that G is abelian.

(b) What can you say about the order of G? Always explain your answers.

(44) Consider the function $f: \mathbb{C}^* \to \mathbb{C}^*$ defined by $f(z) = z^2$, where $\mathbb{C}^*$ denotes the group of non-zero complex numbers under multiplication.

(a) Show that f is a homomorphism.

(b) Find the kernel and image of f.

(c) Describe the quotient of $\mathbb{C}^*$ by the kernel of f.

(d) Is f an isomorphism? Always explain your answers.

(45) Let G be the group of all symmetries of the frieze pattern shown below. (The pattern repeats to fill out an infinite strip in the plane.)

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(a) Copy the pattern and mark
   1. the centres of rotations of G (as small circles),
   2. mirror lines of reflections in G (as solid lines),
   3. axes of glide reflections in G (as dotted lines).

(b) Describe the translation subgroup and the point group for G.

(c) Is the group G abelian? Always explain your answers.

(46) Is the complex matrix

$$\begin{pmatrix} 2+i & 2i \\ -2i & -3+i \end{pmatrix}$$

Hermitian, unitary or normal? Is this matrix diagonalisable? Always explain your answers.

(47) Let W be the complex vector space of all linear transformations $g: \mathbb{C}^3 \to \mathbb{C}$. So the sum and scalar multiplication are given by $(g+h)(v) = g(v) + h(v)$ and $(\lambda g)(v) = \lambda(g(v))$, for $g, h \in W$ and $\lambda \in \mathbb{C}$. Define a complex inner product on W by $(g,h) = g(e_1)\overline{h}(e_1) + g(e_2)\overline{h}(e_2) + g(e_3)\overline{h}(e_3)$, where $\{e_1, e_2, e_3\}$ is the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ for $\mathbb{C}^3$.

Let V be the subspace of W spanned by the transformations f_1, f_2, where $f_1(e_1) = i$, $f_1(e_2) = 1$, $f_1(e_3) = 0$, and $f_2(e_1) = 0$, $f_2(e_2) = -1$, $f_2(e_3) = 2i$. Find the orthogonal complement of V relative to this inner product.

(48) The group A_4 consists of the permutations {(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (234), (243), (134), (143)}.

(a) Find the centraliser and the conjugacy class of the element (12)(34) in the group A_4.

(b) Is the subgroup {(1), (12)(34), (13)(24), (14)(23)} normal in A_4 or not? Always explain your answers.

(49) The set {1,3,5,9,11,13} form a group G under multiplication modulo 14.

(a) Find the orders of the elements of G.

(b) Suppose that there is a homomorphism $\varphi: G \to H$ to the group $\mathbb{Z}/5\mathbb{Z}$. Show that φ maps G to the identity element.

(50) Consider the group G which is the direct product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

(a) List the possible orders of subgroups of G.

(b) Find the orders of all elements of the group G.

(c) Is this group cyclic? Always explain your answers.

2. References

[GH] J.R.J. Groves and C.D. Hodgson, *Notes for 620-297: Group Theory and Linear Algebra*, 2009.

[Ra] A. Ram, *Notes in abstract algebra*, University of Wisconsin, Madison 1994.