Group Theory and linear algebra 16.08.2011

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~~Let F be a field.~~ Lecture 10: Eigenvectors and annihilators.

Let V be a vector space over $\mathbb{C}$

Let $f: V \to V$ be a linear transformation.

The $\underline{\mathbb{C}[t]\text{-module defined by } f}$ is the vector space V with $\mathbb{C}[t]$-action given by

$$p \cdot v = (a_0 + a_1 f + \cdots + a_N f^N) v$$

if $v \in V$ and $p = a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N \in \mathbb{C}[t]$.

<u>Example</u> $V = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \middle| \; a_i \in \mathbb{C} \right\}$ has basis

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$ and the

matrix $B_f = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 & \\ 0 & 0 & 0 & 5 & \end{pmatrix}$ defines $f: V \to V$.

Then

$$(3t + 6t^3) \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \left(3 \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} + 6 \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix}^3 \right) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

$$= \cdots$$

An <u>f-invariant subspace</u>, or <u>$\mathbb{C}[t]$-submodule, of V</u> is a subspace $W \subseteq V$ such that

if $w \in W$ then $fw \in W$.

Let $\lambda \in \mathbb{C}$. The <u>λ-eigenspace of f</u> is

$$V_\lambda = \{ v \in V \mid fv = \lambda v \}.$$

An <u>eigenvector</u> <u>with eigenvalue λ</u> is a vector $v \in V_\lambda$.

<u>Example</u> In our previous example,

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \text{ So } \begin{pmatrix} 7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in V_2.$$

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 0 \end{pmatrix} = 5 \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \text{ So } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in V_5.$$

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \neq 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \text{ So } \cancel{\text{with } c \in \mathbb{C}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \notin V_2.$$

The λ-<u>generalized eigenspace</u> of V is

$$V_\lambda^{gen} = \left\{ v \in V \;\middle|\; \begin{array}{l} \text{there exists } k \in \mathbb{Z}_{\geq 0} \text{ with} \\ (f-\lambda)^k v = 0 \end{array} \right\}$$

In our previos example,

$$f - 2 = \begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 5 & 1 \\ & & & & 5 \end{pmatrix} - 2 \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 3 & 1 \\ & & & & 3 \end{pmatrix}$$

$$(f-2)^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & & \\ & & 0 & & \\ & & & 9 & 6 \\ & & & & 9 \end{pmatrix} \quad \text{and} \quad (f-2)^3 = \begin{pmatrix} 0 & 0 & 0 & & \\ & 0 & 0 & & \\ & & 0 & & \\ & & & 81 & 108 \\ & & & & 81 \end{pmatrix}$$

and

$$(f-2)^3 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{if } a_1, a_2, a_3 \in \mathbb{C}.$$

So

$$V_2^{gen} \supseteq \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; a_1, a_2, a_3 \in \mathbb{C} \right\}$$

The <u>annihilator</u> of V is

$$\mathrm{ann}\, V = \{ p \in \mathbb{C}[t] \mid \text{if } v \in V \text{ then } pv = 0 \}.$$

The <u>minimal polynomial</u> of f is $m \in \mathbb{C}[t]$ such that

$$\mathrm{ann}\, V = m\,\mathbb{C}[t]$$

where $m\,\mathbb{C}[t] = \{ mq \mid q \in \mathbb{C}[t] \} = \{\text{multiples of } m\}$.

In our previous example.

$$(f-2)^3 = \begin{pmatrix} 0&0&0&& \\ 0&0&0&& \\ 0&0&0&& \\ &&&81&108 \\ &&&&81 \end{pmatrix} \quad \text{and} \quad (f-5)^2 = \begin{pmatrix} 9&-6&1&& \\ &9&-6&& \\ &&9&& \\ &&&0&0 \\ &&&0&0 \end{pmatrix}$$

and

$$(f-2)^3(f-5)^2 = \begin{pmatrix} 0&0&0&& \\ 0&0&0&& \\ 0&0&0&& \\ &&&0&0 \\ &&&0&0 \end{pmatrix}$$

So $(t-2)^3(t-5)^2 \in \operatorname{ann} V$.

<u>Theorem</u> Let $f: V \to V$ be a linear transformation.

(a) Let $\lambda \in \mathbb{C}$. Then V_λ is an f-invariant subspace of V.

(b) Let $\lambda \in \mathbb{C}$. Then V_λ^{gen} is an f-invariant subspace of V

(c) If $p_1, p_2 \in \operatorname{ann} V$ then $p_1 + p_2 \in \operatorname{ann} V$.

(d) If $p \in \operatorname{ann} V$ and $q \in \mathbb{C}[t]$ then $pq \in \operatorname{ann} V$.

<u>Proof</u> (a) To show: V_λ is an f-invariant subspace of V.

To show: (aa) If $v_1, v_2 \in V_\lambda$ then $v_1 + v_2 \in V_\lambda$

(ab) If $c \in \mathbb{C}$ and $v \in V_\lambda$ then $cv \in V_\lambda$

(ac) If $v \in V_\lambda$ then $fv \in V_\lambda$.

(aa) Assume $v_1, v_2 \in V_\lambda$

To show: $v_1 + v_2 \in V_\lambda$.

We know: $V_\lambda = \{v \in V \mid fv = \lambda v\}$.

So we know: $fv_1 = \lambda v_1$ and $fv_2 = \lambda v_2$.

To show: $f(v_1+v_2) = \lambda(v_1+v_2)$.

$f(v_1+v_2) = fv_1 + fv_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1+v_2)$.

(ab) Assume $c \in \mathbb{C}$ and $v \in V_\lambda$.

To show: $cv \in V_\lambda$.

To show: $f(cv) = \lambda cv$.

We know: $fv = \lambda v$.

$f(cv) = cf(v)$, since f is a linear transformation,

$= c\lambda v$

$= \lambda cv$, since $\mathbb{C}$ is commutative.

(ac) Assume $v \in V_\lambda$.

To show: $fv \in V_\lambda$

We know: $fv = \lambda v$.

To show: $f(fv) = \lambda fv$

$ffv = f(\lambda v) = \lambda f(v)$, since f is a linear transformation

So V_λ is an f-invariant subspace of V.

(d) Assume $p \in \operatorname{ann} V$ and $q \in \mathbb{C}[t]$.

To show: $pq \in \operatorname{ann} V$.

We know: $\operatorname{ann} V = \{p \in \mathbb{C}[t] \mid \text{if } v \in V \text{ then } pv = 0\}$.

To show: If $v \in V$ then $pqv = 0$.

Assume $v \in V$.

To show: $pqv = 0$.

$$\begin{aligned} pqv &= qpv, \quad \text{since } \mathbb{C}[t] \text{ is commutative,} \\ &= q \cdot 0 = 0 \end{aligned}$$

(c) Assume $p_1, p_2 \in \operatorname{ann} V$.

To show: $p_1 + p_2 \in \operatorname{ann} V$.

To show: If $v \in V$ then $(p_1+p_2)v = 0$.

Assume $v \in V$.

To show: $(p_1+p_2)v = 0$

$$\begin{aligned} (p_1+p_2)v &= p_1 v + p_2 v \\ &= 0 + p_2 v, \quad \text{since } p_1 \in \operatorname{ann} V \\ &= 0 + 0, \quad \text{since } p_2 \in \operatorname{ann} V \\ &= 0. \end{aligned}$$