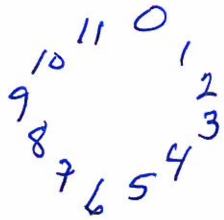
$\frac{\mathbb{C}[t]}{m\,\mathbb{C}[t]}$ Lecture 11 and multiplication by t. 17.08.2011

Group Theory Linear algebra

In the same way that

$\mathbb{Z}/m\mathbb{Z}$ is $\mathbb{Z}$ with $m=0$



$\frac{\mathbb{C}[t]}{m\,\mathbb{C}[t]}$ is $\mathbb{C}[t]$ with $m=0$.

Example Let $m=(t-2)^2 = t^2-4t+4$.

Then, in $\frac{\mathbb{C}[t]}{(t-2)^2\mathbb{C}[t]}$, $t^2-4t+4=0$, $t^2=4t-4$,

and $t^3+7t^2+5t+3 = t\cdot t^2+7t^2+5t+3$

$$= t(4t-4)+7(4t-4)+5t+3$$
$$= 4t^2-4t+28t-28+5t+3$$
$$= 4(4t-4)-4t+28t+5t-25$$
$$= 16t-16+29t-25 = 45t-41$$

Any polynomial in $\frac{\mathbb{C}[t]}{(t-2)^2\mathbb{C}[t]}$ is a linear combination of 1 and t.

$B=\{1,t\}$ is a basis of $\frac{\mathbb{C}[t]}{(t-2)^2\mathbb{C}[t]}$

$C=\{1,t-2\}$ is another basis of $\frac{\mathbb{C}[t]}{(t-2)^2\mathbb{C}[t]}$

and the change of basis matrix from B to C is

$$P = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Let $V = \dfrac{\mathbb{C}[t]}{(t-2)^2\mathbb{C}[t]}$ and let $f: V \to V$ be the linear transformation given by

$$f(p) = tp.$$

For example: $f(45t-41) = t(45t-41)$

$$= 45t^2 - 41 = 45(4t-4) - 41$$
$$= 180t - 180 - 41 = 180t - 221.$$

The matrix of f with respect to B is

$$B_f = \begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix} \text{ since } f(t) = t^2 = 4t-4.$$

The matrix of f with respect to C is

$$C_f = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

<u>Example</u> $V = \dfrac{\mathbb{C}[t]}{(t-2)(t-3)\mathbb{C}[t]}$ has $(t-2)(t-3) = 0$,

so that $t^2 - 5t + 6 = 0$ and $t^2 = 5t - 6$.

V has bases

$B = \{1, t\}$ and

$C = \{t-2, -t+3\}$

Let $f: V \to V$ be the linear transformation given by

$$f(p) = tp.$$

The matrix of f with respect to B is

$$B_f = \begin{pmatrix} 0 & -6 \\ 1 & 5 \end{pmatrix}$$

The matrix of f with respect to C is

$$C_f = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

since

$$f(t-2) = t^2 - 2t = 5t - 6 - 2t = 3t - 6 = 3(t-2)$$

$$f(-t+3) = -t^2 + 3t = -(5t-6) + 3t = -2t + 6 = 2(-t+3)$$

<u>Example</u> $V = \dfrac{\mathbb{C}[t]}{(t-3)\mathbb{C}[t]} \oplus \dfrac{\mathbb{C}[t]}{(t-2)\mathbb{C}[t]}$

$$= \left\{ (u, w) \;\middle|\; u \in \frac{\mathbb{C}[t]}{(t-3)\mathbb{C}[t]},\ w \in \frac{\mathbb{C}[t]}{(t-2)\mathbb{C}[t]} \right\}$$

and $t \cdot (u, w) = (tu, tw)$.

V has basis

$$C = \{ (1,0), (0,1) \}$$

with

$$t(1,0) = (t, 0) = (3, 0) = 3(1,0)$$

$$t(0,1) = (0, t) = (0, 2) = 2(0,1)$$

since $t = 3$ in $\dfrac{\mathbb{C}[t]}{(t-3)\mathbb{C}[t]}$ and $t = 2$ in $\dfrac{\mathbb{C}[t]}{(t-2)\mathbb{C}[t]}$

So the matrix of $f: V \to V$ given by

$$f(u,w) = t(u,w),$$

is

$$C_f = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

The matrix of f with respect to the basis

$$\mathcal{B} = \{(1,1), (t,t)\} = \{(1,1), (3,2)\}$$

is

$$B = \begin{pmatrix} 0 & -6 \\ 1 & 5 \end{pmatrix}.$$

since $t(3,2) = (3t, 2t) = (9,4) = -6(1,1) + 5(3,2)$

The function

$$\Phi : \frac{\mathbb{C}[t]}{(t-2)(t-3)\mathbb{C}[t]} \longrightarrow \frac{\mathbb{C}[t]}{(t-3)\mathbb{C}[t]} \oplus \frac{\mathbb{C}[t]}{(t-2)\mathbb{C}[t]}$$

given by $\Phi(1) = (1,1)$ and $\Phi(t) = (t,t) = (3,2)$

is a linear transformation such that

if $v \in \dfrac{\mathbb{C}[t]}{(t-2)(t-3)\mathbb{C}[t]}$ then $\Phi(tv) = t\,\Phi(v)$

<u>HW</u>: Show that Φ is bijective.