Group Theory and linear algebra Lecture 12, 19.08.2011

Jordan Normal form

Let A be an $n \times n$ matrix

If $I = \{i_1, \dots, i_m\}$ is a subset of $\{1, 2, \dots, n\}$

and $J = \{j_1, \dots, j_m\}$ another subset of $\{1, 2, \dots, n\}$

the (I, J) minor of $t - A$ is

$$\det(t-A)_{IJ} = \det \begin{pmatrix} (t-A)_{i_1 j_1} & (t-A)_{i_1 j_2} & \cdots & (t-A)_{i_1 j_m} \\ \vdots & & \ddots & \vdots \\ (t-A)_{i_m j_1} & (t-A)_{i_m j_2} & \cdots & (t-A)_{i_m j_m} \end{pmatrix}$$

If I and J have m elements $\det((t-A)_{IJ})$ is an m^{th} order minor of $t-A$.

The gcd of the m^{th} order minors of $t-A$ is

$$d_m(t) = \gcd\left\{ \det(t-A)_{IJ} \;\middle|\; \begin{array}{c} \det(t-A)_{IJ} \text{ is an } m^{th} \text{ order} \\ \text{minor of } t-A \end{array} \right\}$$

The similarity invariants of A, or invariant factors of $t-A$, are monic polynomials $q_1(t), \dots, q_n(t)$ such that

$$d_m(t) = q_1(t)\, q_2(t) \cdots q_m(t), \quad \text{for } m = 1, 2, \dots, n$$

(i.e. $q_m(t) = \dfrac{d_m(t)}{d_{m-1}(t)}$).

The characteristic polynomial of A is $d_n(t)$.

Theorem The minimal polynomial of A is $q_n(t)$.

A <u>Jordan block</u> of type $(t-\lambda)^e$ is the $e \times e$ matrix

$$\begin{pmatrix} \lambda & & & & & \\ 1 & \lambda & & & 0 & \\ & 1 & \lambda & & & \\ & & & \ddots & & \\ & 0 & & & \lambda & \\ & & & & 1 & \lambda \end{pmatrix}$$

Factor the $q_1(t), q_2(t), \ldots, q_n(t)$:

$$q_1(t) = (t-\lambda_1)^{e_{11}}(t-\lambda_2)^{e_{12}} \cdots (t-\lambda_k)^{e_{1k}}$$

$$q_2(t) = (t-\lambda_1)^{e_{21}}(t-\lambda_2)^{e_{22}} \cdots (t-\lambda_k)^{e_{2k}}$$

$$\vdots$$

$$q_n(t) = (t-\lambda_1)^{e_{n1}}(t-\lambda_2)^{e_{n2}} \cdots (t-\lambda_k)^{e_{nk}}$$

<u>Theorem</u> There exists an invertible matrix P such that

$$PAP^{-1} = \begin{pmatrix} J_{11} & & & & & & & \\ & J_{12} & & & & & 0 & \\ & & \ddots & & & & & \\ & & & J_{1k} & & & & \\ & & & & J_{21} & & & \\ & & & & & J_{22} & & \\ & & & & & & \ddots & \\ & 0 & & & & & J_{n,k-1} & \\ & & & & & & & J_{n,k} \end{pmatrix}$$

where $J_{m\ell}$ is a Jordan block of type $(t-\lambda_\ell)^{e_{m\ell}}$

<u>Example</u> If $A = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 0 & 1 \\ 2 & 6 & 3 \end{pmatrix}$ then $t - A = \begin{pmatrix} t-2 & 1 & 3 \\ 5 & t & 1 \\ 2 & 6 & t-3 \end{pmatrix}$

<u>Subsets of $\{1,2,3\}$</u>

If I and J have 3 elements then $I = \{1,2,3\} = J$ and $(t-A)_{IJ} = t-A$ and $d_3 = \det(t-A)$. So

$$d_3(t) = \det(t-A) = (t-2)(t^2-3t-6) - (5t-15-2) + 3(30-2t)$$
$$= t^3 - 3t^2 - 6t - 2t^2 + 6t + 12 - 5t + 17 + 90 - 6t$$
$$= t^3 - 5t^2 - 11t + 119$$

If I and J have 1 element then $(t-A)_{IJ}$ is a single entry of $t-A$. So

$$d_1(t) = \gcd\{t-2, 1, 3, 5, t, 1, 2, 6, t-3\} = 1.$$

If I and J have 2 elements then

I is $\{1,2\}$ or $\{1,3\}$ or $\{2,3\}$ and
J is $\{1,2\}$ or $\{1,3\}$ or $\{2,3\}$ and

$$d_2(t) = \gcd\left\{ \begin{matrix} \det\begin{pmatrix} t-2 & 1 \\ 5 & t \end{pmatrix}, \det\begin{pmatrix} t-2 & 3 \\ 5 & 1 \end{pmatrix}, \det\begin{pmatrix} 1 & 3 \\ t & 1 \end{pmatrix}, \\ \det\begin{pmatrix} t-2 & 1 \\ 2 & 6 \end{pmatrix}, \det\begin{pmatrix} t-2 & 3 \\ 2 & t-3 \end{pmatrix}, \det\begin{pmatrix} 1 & 3 \\ 6 & t-3 \end{pmatrix}, \\ \det\begin{pmatrix} 5 & t \\ 2 & 6 \end{pmatrix}, \det\begin{pmatrix} 5 & 1 \\ 2 & t-3 \end{pmatrix}, \det\begin{pmatrix} t & 1 \\ 6 & t-3 \end{pmatrix} \end{matrix} \right\}$$

$$= \gcd\{t^2-2t-5, t-10, 1-3t, \ldots, 5t-17, t^2-3t-6\} = 1.$$

Wolfram alpha tells us that

$t^3 - 5t^2 - 11t + 119$

$= (t + 4.23445)(t - (4.61723 + 2.60462i))$
$(t - (4.61723 - 2.60462i))$

Since $d_1(t) = q_1(t)$, $d_2(t) = q_1(t) q_2(t)$, $d_3(t) = q_1(t) q_2(t) q_3(t)$ and

$d_1(t) = 1$ | $q_1(t) = 1$
$d_2(t) = 1$ then $q_2(t) = 1$
$d_3(t) = t^3 - 5t^2 - 11t + 119$, | $q_3(t) = t^3 - 5t^2 - 11t + 119$

and the factorization of $q_1(t)$, $q_2(t)$ and $q_3(t)$ are

$q_1(t) = (t - \alpha_1)^0 (t - \alpha_2)^0 (t - \alpha_3)^0$

$q_2(t) = (t - \alpha_1)^0 (t - \alpha_2)^0 (t - \alpha_3)^0$

$q_3(t) = (t - \alpha_1)^1 (t - \alpha_2)^1 (t - \alpha_3)^1$

where

$\alpha_1 = 4.23445$, $\alpha_2 = 4.61723 + i2.60462$,
$\alpha_3 = 4.61723 - i2.60462$.

The Jordan normal form theorem then says that there exists P such that

$$PAP^{-1} = P \begin{pmatrix} 2 & 1 & 3 \\ 5 & 0 & 1 \\ 2 & 6 & 3 \end{pmatrix} P^{-1} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

with $\alpha_1, \alpha_2, \alpha_3$ as above.