Lecture 13 23.08.2011 Group Theory and linear algebra 

**Proposition** Let $V$ be a vector space. Let $B=\{b_1,\dots,b_k\}$ and $C=\{c_1,\dots,c_\ell\}$ be bases of $V$. ~~Let~~ Then $k=\ell$.

**Proof** Proof by contradiction.

Assume $k<\ell$.

Let $C'=\{c_2,\dots,c_\ell\}$.

Then there exists $b_j\in B$ such that $b_j\notin \text{span}(C')$.

**Proof** If all $b_1,\dots,b_k\in\text{span}(C')$ then

$\text{span}(B)\subseteq\text{span}(C')$ and $V\subseteq\text{span}(C')$.

Since $c_1\in V$ then $c_1\in\text{span}(C')$.

So $c_1=\gamma_2c_2+\dots+\gamma_\ell c_\ell$ for some $\gamma_2,\dots,\gamma_\ell\in\mathbb{F}$.

So $0=-c_1+\gamma_2c_2+\dots+\gamma_\ell c_\ell$.

This is a contradiction to the linear independence of $C$.

So there exists $b_j$ such that $b_j\notin\text{span}(C')$. //

Reindex $B$ so that $b_1\notin\text{span}(C')$ and let $C_1=\{b_1,c_2,c_3,\dots,c_\ell\}$.

**Claim** $C_1$ is a basis of $V$.

**Proof of claim:**

To show: (a) $C_1$ is linearly independent

(b) $\text{span}(C_1)=V$.

(a) To show: If $\alpha_1 b_1 + \alpha_2 c_2 + \cdots + \alpha_\ell c_\ell = 0$ then
$\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_\ell = 0$.

Assume $\alpha_1 b_1 + \alpha_2 c_2 + \cdots + \alpha_\ell c_\ell = 0$.

<u>Case 1</u> $\alpha_1 \neq 0$.

Then $b_1 = -\frac{\alpha_2}{\alpha_1} c_2 - \frac{\alpha_3}{\alpha_1} c_3 - \cdots - \frac{\alpha_\ell}{\alpha_1} c_\ell$.

So $b_1 \in \mathrm{span}(C')$, which is a contradiction to the choice of $b_1$

So $\alpha_1 = 0$.

<u>Case 2</u> $\alpha_1 = 0$.

Then $\alpha_2 c_2 + \cdots + \alpha_\ell c_\ell = 0$.

Since $C$ is linearly independent
$\alpha_2 = 0, \alpha_3 = 0, \ldots, \alpha_\ell = 0$. //

(b) To show: $V = \mathrm{span}\{b_1, c_2, \ldots, c_\ell\}$.

To show: $V \subseteq \mathrm{span}\{b_1, c_2, \ldots, c_\ell\}$.

To show: $\mathrm{span}\{c_1, c_2, \ldots, c_\ell\} \subseteq \mathrm{span}\{b_1, c_2, \ldots, c_\ell\}$.

To show: $c_1 \in \mathrm{span}\{b_1, c_2, \ldots, c_\ell\}$.

Since $C$ is a basis there exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{F}$ with
$b_1 = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_\ell c_\ell$.

<u>Case 1</u> $\alpha_1 = 0$.

Then $b_1 = \alpha_2 c_2 + \cdots + \alpha_\ell c_\ell$ and $b_1 \in \mathrm{span}(C')$.

This is a contradiction to the choice of $b_1$.

So $\alpha_1 \neq 0$.

Case 2 $\alpha_1 \neq 0$

Then $c_1 = +\alpha_1^{-1}(b_1 - \alpha_2 c_2 - \cdots - \alpha_\ell c_\ell)$

So $c_1 \in \operatorname{span}\{b_1, c_2, \ldots, c_\ell\}$. //

So $C_1 = \{b_1, c_2, \ldots, c_\ell\}$ is a basis of $V$.

Let $C_1' = \{b_1, c_3, \ldots, c_\ell\}$ and let $b_j \in B$ such that $b_j \notin \operatorname{span}(C_1')$.

Reindex $B$ so that $b_2 \notin \operatorname{span}(C_1')$.

Let $C_2 = \{b_1, b_2, c_3, \ldots, c_\ell\}$.

Then, by a proof as for $C_1$ above, $C_2$ is a basis of $V$.

Continue this process to obtain

$C_k = \{b_1, b_2, \ldots, b_k, c_{k+1}, c_{k+2}, \ldots, c_\ell\}$ which is a basis of $V$.

Since $B$ is a basis of $V$, then

$c_{k+1} = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_k b_k$, for some $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$.

So $0 = \alpha_1 b_1 + \cdots + \alpha_k b_k - c_{k+1}$.

This contradicts the linear independence of $C_k$.

So $k \nless \ell$.

A similar argument shows $\ell \nless k$.

So $k = \ell$. //

Proposition

Let $V$ be a vector space and let $f: V \to V$ be a linear transformation.

Let $m(t)$ be the minimal polynomial of $f$.

Assume that

$$m(t) = p(t)q(t) \text{ with } \gcd(p,q) = 1.$$

Let $k(t)$ and $l(t)$ be such that

$$1 = p(t)k(t) + q(t)l(t)$$

Let

$$U = p(f)k(f)V \quad \text{and} \quad W = q(f)l(f)V.$$

Then $$V = U \oplus W.$$

Example Let $A = \begin{pmatrix} 2 & 0 & & \\ 1 & 2 & & \\ & & 3 & 0 \\ & & 1 & 3 \end{pmatrix}$

The minimal polynomial of $A$ is

$$m(t) = (t-2)^2(t-3)^2 \text{ and } p(t) = (t-2)^2,\ q(t) = (t-3)^2.$$

$$p(t) = t^2 - 4t + 4 \text{ and } q(t) = t^2 - 6t + 9$$

$$t^2 - 4t + 4 = (t^2 - 6t + 9) + (2t - 5)$$

$$t^2 - 6t + 9 = \cancel{t^2 - 4t + 4}\ (2t-5)\left(\tfrac{1}{2}t - \tfrac{7}{4}\right) + \tfrac{1}{4}$$

since

$$t^2-4t+4=(t^2-6t+9)+(2t-5)$$

$$\begin{array}{r|l} & \tfrac{1}{2}t-\tfrac{7}{4} \\ \hline 2t-5 & t^2-6t+9 \\ & t^2-\tfrac{5}{2}t \\ \hline & -\tfrac{7}{2}t+9 \\ & -\tfrac{7}{2}t+\tfrac{35}{4} \\ \hline & \tfrac{1}{4} \end{array}$$

So

$$t^2-6t+9=\left(\tfrac{1}{2}t-\tfrac{7}{4}\right)(2t-5)+\tfrac{1}{4}$$

So

$$\begin{aligned} 1 &= 4(t^2-6t+9)-(2t-7)(2t-5) \\ &= 4(t^2-6t+9)-(2t-7)\big((t^2-4t+4)-(t^2-6t+9)\big) \\ &= (2t-3)(t^2-6t+9)-(2t-7)(t^2-4t+4) \\ &= (2t-3)(t-3)^2-(2t-7)(t-2)^2 \end{aligned}$$

Then

$$(2A-3)(A-3)^2=\begin{pmatrix} +1 & 0 & & \\ 2 & +1 & & \\ & & 3 & 0 \\ & & 2 & 3 \end{pmatrix}\begin{pmatrix} -1 & & & \\ 1 & -1 & & \\ & & 0 & \\ & & 1 & 0 \end{pmatrix}^2$$

$$=\begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ & & 3 & \\ & & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & & \\ -2 & 1 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

and

$$-(2A-7)(A-2)^2=\begin{pmatrix} 3 & 0 & & \\ -2 & 3 & & \\ & & 1 & 0 \\ & & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 1 & 0 \\ & & 1 & 1 \end{pmatrix}^2=\begin{pmatrix} 3 & 0 & & \\ -2 & 3 & & \\ & & 1 & 0 \\ & & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 1 & 0 \\ & & 2 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & & \\ \multicolumn{2}{c}{0} & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$$