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Lecture 14, Group Theory and linear algebra

The Cayley-Hamilton theorem

Let A be a matrix in Jordan normal form.

Let $m_A(t)$ be the minimal polynomial of A

Let $c_A(t)$ be the characteristic polynomial of A.

$m_A(t)$ is minimal degree such that $m(A)=0$.

$c_A(t) = \det(t-A)$.

Example

If $J = \begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \lambda & \\ & & 1 & \lambda \end{pmatrix}$ is a Jordan block of type $(t-\lambda)^4$ then

$$J-\lambda = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{pmatrix}, \quad (J-\lambda)^2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0 \end{pmatrix}, \quad (J-\lambda)^3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}$$

and $(J-\lambda)^4 = 0$

If $m_A(t) = (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\cdots(t-\lambda_k)^{m_k}$ and

$c_A(t) = (t-\lambda_1)^{c_1}(t-\lambda_2)^{c_2}\cdots(t-\lambda_k)^{c_k}$

then m_j is the maximal size of a Jordan block with eigenvalue λ_j in A

c_j is the sum of the sizes of the Jordan blocks of eigenvalue λ_j in A.

So, finding the minimal polynomial and characteristic polynomial of a matrix in Jordan normal form is easy.

<u>Example</u> $A = \begin{pmatrix} 2 &&&&&&&&&&& \\ 1 & 2 &&&&&&&&&& \\ && 2 &&&&&&&&& \\ &&& 2 &&&&&&&& \\ &&& 1 & 2 &&&&&&& \\ &&&& 1 & 2 &&&&&& \\ &&&&&& 3 &&&&& \\ &&&&&& 1 & 3 &&&& \\ &&&&&&&& 3 &&& \\ &&&&&&&& 1 & 3 && \\ &&&&&&&&& 1 & 3 & \\ &&&&&&&&&& 1 & 3 \end{pmatrix}$

Then $m(t) = (t-2)^3(t-3)^4$ and $c(t) = (t-2)^6(t-3)^6$

since the sizes of Jordan blocks of eigenvalue 2 are 2, 1, 3 (with maximum 3)

and the sizes of Jordan blocks of eigenvalue 3 are 2, 4 (with maximum 4).

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Let Q be an $n \times n$ matrix.

The Jordan normal form theorem says that:

There exists P such that

$$PQP^{-1} = A \text{ is in Jordan normal form.}$$

Let us compare $m_Q(t)$ and $m_A(t)$

and compare $c_Q(t)$ and $c_A(t)$.

$c_Q(t) = \det(t - Q)$

$$c_A(t) = \det(t - A) = \det(t - PQP^{-1})$$
$$= \det(tPP^{-1} - PQP^{-1}) = \det(P(t-Q)P^{-1})$$
$$= \det(P)\det(t-Q)\det(P^{-1}) = \det(t-Q) = c_Q(t).$$

If $l(t)$ is a polynomial then

$$l(t) = l_0 + l_1 t + l_2 t^2 + \cdots + l_d t^d, \quad \text{and}$$

$$l(Q) = l_0 + l_1 Q + l_2 Q^2 + \cdots + l_d Q^d, \quad \text{and}$$

$$\begin{aligned} l(A) &= l_0 + l_1 A + l_2 A^2 + \cdots + l_d A^d \\ &= l_0 + l_1 PQP^{-1} + l_2 (PQP^{-1})^2 + \cdots + l_d (PQP^{-1})^d. \end{aligned}$$

Since

$$(PQP^{-1})^2 = PQP^{-1}PQP^{-1} = PQ^2P^{-1},$$

$$(PQP^{-1})^3 = PQP^{-1}PQP^{-1}PQP^{-1} = PQ^3P^{-1}, \quad \ldots,$$

then

$$\begin{aligned} l(A) &= l_0 + l_1 PQP^{-1} + l_2 PQ^2P^{-1} + \cdots + l_d PQ^dP^{-1} \\ &= P(l_0 + l_1 Q + l_2 Q^2 + \cdots + l_d Q^d)P^{-1} \\ &= P\, l(Q) P^{-1} \end{aligned}$$

and $l(Q) = P^{-1} l(A) P$.

So if $l(Q) = 0$ then $l(A) = 0$ and
if $l(A) = 0$ then $l(Q) = 0$.

$m_Q(t)$ is the smallest degree monic polynomial such that $m_Q(Q) = 0$.

$m_A(t)$ is the smallest degree monic polynomial such that $m_A(A) = 0$.

<u>The point</u>: If $PQP^{-1} = A$, then $m_Q(t) = m_A(t)$.
If $PQP^{-1} = A$, then $c_Q(t) = c_A(t)$.

<u>The Cayley-Hamilton Theorem</u> Let Q be an $n \times n$ matrix and let

$$c_Q(t) = \det(t - Q)$$

be the characteristic polynomial of Q. Then

$$c_Q(Q) = 0.$$