Lecture 15 Group Theory and linear algebra

Let V be a vector space over ~~$\mathbb{R}$ or~~ $\mathbb{C}$.

A *positive definite* *Hermitian* *form*, or *inner product*, on V is a function

$$\begin{aligned} V \times V &\to \mathbb{C} \\ (v_1, v_2) &\mapsto \langle v_1, v_2 \rangle \end{aligned} \qquad \text{such that}$$

(a) If $v_1, v_2 \in V$ then $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$,

(b) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, v_3 \in V$ then

$$\langle c_1 v_1 + c_2 v_2, v_3 \rangle = c_1 \langle v_1, v_3 \rangle + c_2 \langle v_2, v_3 \rangle$$

(c) If $v \in V$ then $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$

(d) If $v \in V$ and $\langle v, v \rangle = 0$ then $v = 0$.

Let $v \in V$. The *length* of v is

$$\|v\| = \sqrt{\langle v, v \rangle} \quad \text{in } \mathbb{R}_{\geq 0}.$$

so that $\| \ \| : V \to \mathbb{R}_{\geq 0}$ is given by $\|v\| = \sqrt{\langle v, v \rangle}$.

Let $u, w \in V$. The elements u, w are *orthogonal* if $\langle u, w \rangle = 0$.

An *orthonormal* *basis* of V is a basis $\{v_1, \ldots, v_k\}$ of V such that

$$\langle v_i, v_j \rangle = \delta_{ij}, \quad \text{where } \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Let V be a vector space over $\mathbb{C}$ with a positive definite Hermitian form
$$\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}.$$

~~that~~ Let $B = \{b_1, b_2, \ldots, b_k\}$ be a basis of V.

The matrix of $\langle\ ,\ \rangle$ with respect to B is
$$A = (\langle b_i, b_j \rangle).$$

If $v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_k b_k$ and
$w = \gamma_1 b_1 + \gamma_2 b_2 + \cdots + \gamma_k b_k$

then
$$\langle v, w \rangle = \langle \alpha_1 b_1 + \cdots + \alpha_k b_k, \gamma_1 b_1 + \gamma_2 b_2 + \cdots + \gamma_k b_k \rangle$$
$$= \alpha_1 \bar{\gamma}_1 \langle b_1, b_1 \rangle + \alpha_1 \bar{\gamma}_2 \langle b_1, b_2 \rangle + \cdots + \alpha_1 \bar{\gamma}_k \langle b_1, b_k \rangle$$
$$+ \alpha_2 \bar{\gamma}_1 \langle b_2, b_1 \rangle + \cdots$$
$$\cdots + \alpha_k \bar{\gamma}_k \langle b_k, b_k \rangle$$
$$= \sum_{i,j=1}^{k} \alpha_i \bar{\gamma}_j \langle b_i, b_j \rangle = \sum_{i,j=1}^{k} \alpha_i \langle b_i, b_j \rangle \bar{\gamma}_j.$$
$$= (\alpha_1, \alpha_2, \ldots, \alpha_k) \begin{pmatrix} & & \\ & \langle b_i, b_j \rangle & \\ & & \end{pmatrix} \begin{pmatrix} \bar{\gamma}_1 \\ \bar{\gamma}_2 \\ \vdots \\ \bar{\gamma}_k \end{pmatrix}$$
$$= v^t A \bar{w}$$

<u>Note</u>: Since $\overline{\langle b_i, b_j \rangle} = \langle b_j, b_i \rangle$,
$\bar{A}_{ij} = A_{ji}$. So $\bar{A}^t = A$.

Creating orthonormal bases: Gram-Schmidt

Example Let V be a vector space with basis $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\langle\, , \rangle : V \times V \to \mathbb{C}$ having matrix

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 3 \end{pmatrix} \text{ with respect to } \mathcal{B}.$$

Then $\langle b_1, b_1 \rangle = 1$. Let $v_1 = b_1$. So $\langle v_1, v_1 \rangle = 1$.

Then $\langle b_2, v_1 \rangle = \langle b_2, b_1 \rangle = 0$ and $\langle b_2, b_2 \rangle = 2$. Let $v_2 = \frac{1}{\sqrt{2}} b_2$

so that $\langle v_2, v_1 \rangle = \langle \frac{1}{\sqrt{2}} b_2, b_1 \rangle = \frac{1}{\sqrt{2}} \langle b_2, b_1 \rangle = 0$ and

$\langle v_2, v_2 \rangle = \langle \frac{1}{\sqrt{2}} b_2, \frac{1}{\sqrt{2}} b_2 \rangle = \frac{1}{2} \langle b_2, b_2 \rangle = \frac{1}{2} \cdot 2 = 1$.

Now $\langle b_3, v_1 \rangle = \langle b_3, b_1 \rangle = -2$

$\langle b_3, v_2 \rangle = \frac{1}{\sqrt{2}} \langle b_3, b_2 \rangle = \frac{1}{\sqrt{2}}$. Let $b_3' = b_3 - (-2) v_1 - \frac{1}{\sqrt{2}} v_2$.

Then $\langle b_3', v_1 \rangle = \langle b_3 - (-2) v_1 - \frac{1}{\sqrt{2}} v_2, v_1 \rangle = \langle b_3, v_1 \rangle + 2 \langle v_1, v_1 \rangle - 0 = -2 + 2 = 0$

$\langle b_3', v_2 \rangle = \langle b_3 - (-2) v_1 - \frac{1}{\sqrt{2}} v_2, v_2 \rangle = \langle b_3, v_2 \rangle + 2 \cdot 0 - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$.

$\langle b_3', b_3' \rangle = \langle b_3', b_3 - (-2) v_1 - \frac{1}{\sqrt{2}} v_2 \rangle = \langle b_3', b_3 \rangle + 0 + 0$

$= \langle b_3 - (-2) v_1 - \frac{1}{\sqrt{2}} v_2, b_3 \rangle = 3 + 2(-2) - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = -1 - \frac{1}{2} = -\frac{3}{2}$.

Let $v_3 = \frac{1}{\sqrt{-\frac{3}{2}}} b_3' = \frac{\sqrt{2}}{\sqrt{3}\, i} b_3' = \frac{-\sqrt{2}\, i}{\sqrt{3}} b_3' = -\frac{\sqrt{2}}{3} i\, b_3 - \frac{2\sqrt{2}}{\sqrt{3}} i\, v_1 + \frac{i}{\sqrt{3}} v_2$.

Then $\langle v_3, v_1 \rangle = \langle -\frac{\sqrt{2}\, i}{\sqrt{3}} b_3', v_1 \rangle = 0$ and $\langle v_3, v_2 \rangle = \langle -\frac{\sqrt{2}\, i}{\sqrt{3}} b_3', v_2 \rangle = 0$

and $\langle v_3, v_3 \rangle = \langle \frac{1}{\sqrt{-\frac{3}{2}}} b_3', \frac{1}{\sqrt{-\frac{3}{2}}} b_3' \rangle = \frac{-1}{\left(-\frac{3}{2}\right)} \left(-\frac{3}{2}\right) = -1$.

<u>Example</u> Let $P_2(\mathbb{C}) = \{a_1 + a_2 x \mid a_1, a_2 \in \mathbb{C}\}$

with $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$.

Then $B = \{1, x\}$ is a basis of $P_2(\mathbb{C})$.

$$\langle 1, 1\rangle = \int_0^1 dx = x\Big|_0^1 = 1$$

$$\langle 1, x\rangle = \int_0^1 \bar{x}\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\langle x, x\rangle = \int_0^1 x\bar{x}\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

So the matrix of $\langle , \rangle$ with respect to the basis $\{1, x\}$ is

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

Let $v_1 = 1$. Then $\langle v_1, v_1\rangle = \langle 1, 1\rangle = 1$.

$$\langle x, v_1\rangle = \langle x, 1\rangle = \frac{1}{2}.$$

Let $b_2' = x - \langle x, v_1\rangle v_1 = x - \frac{1}{2}v_1$.

Then $\langle b_2', v_1\rangle = \langle x - \frac{1}{2}v_1, v_1\rangle = \langle x, v_1\rangle - \frac{1}{2}\langle v_1, v_1\rangle = \frac{1}{2} - \frac{1}{2} = 0$.

$$\langle b_2', b_2'\rangle = \langle b_2', x - \tfrac{1}{2}v_1\rangle = \langle b_2', x\rangle - 0$$

$$= \langle x - \tfrac{1}{2}v_1, x\rangle = \frac{1}{3} - \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{12}$$

Let $v_2 = \frac{1}{1/\sqrt{12}} b_2' = \sqrt{12}\, b_2' = 2\sqrt{3}(x - \frac{1}{2}v_1) = 2\sqrt{3}x - \sqrt{3}$

Then $\langle v_2, v_2\rangle = \langle \frac{1}{\sqrt{1/12}} b_2', \frac{1}{\sqrt{1/12}} b_2'\rangle = \frac{1}{1/12}\langle b_2', b_2'\rangle = \frac{1/12}{1/12} = 1$

and $\langle v_2, v_1\rangle = \langle \frac{1}{\sqrt{1/12}} b_2', v_1\rangle = \frac{1}{\sqrt{1/12}}\cdot 0 = 0$. So $\{v_1, v_2\}$ is an orthonormal basis