## Lecture 16: Group Theory and linear algebra   Inner products: adjoints and complements.   (1)

Let $V$ be a vector space over $\mathbb{C}$.

Let $\langle\, ,\rangle : V\times V \to \mathbb{C}$ be a positive definite Hermitian form.

Let $W$ be a subspace of $V$.

The orthogonal complement to $W$ is

$$W^\perp = \{v\in V \mid \text{if } w\in W \text{ then } \langle v,w\rangle = 0\}$$

**Proposition** (a) $W^\perp$ is a subspace of $V$

(b) $V = W \oplus W^\perp$.

**Proof** (a) To show: (aa) If $u_1,u_2\in W^\perp$ then $u_1+u_2\in W^\perp$

(ab) If $u\in W^\perp$ and $c\in\mathbb{C}$ then $cu\in W^\perp$.

(aa) Assume $u_1,u_2\in W^\perp$.

To show: $u_1+u_2\in W^\perp$.

To show: If $w\in W$ then $\langle u_1+u_2, w\rangle = 0$.

Assume $w\in W$.

To show: $\langle u_1+u_2, w\rangle = 0$.

$$\begin{aligned}\langle u_1+u_2, w\rangle &= \langle u_1,w\rangle + \langle u_2,w\rangle\\ &= 0+0, \text{ since } u_1,u_2\in W^\perp.\\ &= 0\end{aligned}$$

(ab) Assume $u\in W^\perp$ and $c\in\mathbb{C}$.

To show: $cu\in W^\perp$

To show: If $w\in W$ then $\langle cu, w\rangle = 0$.

Assume $w\in W$.

To show: $\langle cu, w\rangle = 0$.

$\langle cu, w\rangle = c\langle u, w\rangle = c\cdot 0$, since $u \in W^\perp$
$= 0$

(b) To show: $V = W \oplus W^\perp$

To show: (ba) $W \cap W^\perp = \{0\}$

(bb) $W + W^\perp = V$.

Choose an orthonormal basis of $W$ (by Gram-Schmidt)
Extend this to an orthonormal basis of all of $V$ (by more Gram-Schmidt).

$$\underbrace{\{\overbrace{b_1, b_2, \ldots, b_k}^{\text{basis of } W}, b_{k+1}, b_{k+2}, \ldots, b_{k+\ell}\}}_{\text{basis of } V}$$

Then $\{b_{k+1}, b_{k+2}, \ldots, b_{k+\ell}\}$ is an orthonormal basis of $W^\perp$:

If $w = c_1 b_1 + \cdots + c_k b_k \in W$ then

$$\begin{aligned}\langle b_{k+i}, w\rangle &= \langle b_{k+i}, c_1 b_1 + \cdots + c_k b_k\rangle \\ &= c_1\langle b_{k+i}, b_1\rangle + \cdots + c_k\langle b_{k+i}, b_k\rangle \\ &= c_1\cdot 0 + \cdots + c_k\cdot 0 \\ &= 0,\end{aligned}$$

so that $b_{k+i} \in W^\perp$.

## Adjoints

Let $V$ be a vector space over $\mathbb{C}$ and
$\langle , \rangle : V \times V \to \mathbb{C}$ a positive definite Hermitian form.

Let $f: V \to V$ be a linear transformation

The <u>adjoint of $f$</u> is a linear transformation
$f^*: V \to V$ such that
if $u, w \in V$ then $\langle f(u), w \rangle = \langle u, f^*(w) \rangle$.

The linear transformation $f: V \to V$ is

- <u>self adjoint</u>, or <u>Hermitian</u>, if $f$ satisfies
$$f = f^*;$$
- an <u>isometry</u>, or <u>unitary</u>, if $f$ satisfies
$$f^* f = 1,$$
- <u>normal</u>, if $f$ satisfies
$$f^* f = f f^*.$$

<u>Theorem</u> Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and $\langle , \rangle : V \times V \to \mathbb{C}$ a positive definite Hermitian form. Let $f: V \to V$ be a linear transformation and $\mathcal{B} = \{b_1, \ldots, b_k\}$ an orthonormal basis of $V$. Then

$$\mathcal{B}_{f^*} = (\overline{\mathcal{B}_f})^t.$$

If $A$ is a matrix with $(i,j)$-entry $A_{ij}$ then
$A^t$ is a matrix with $(i,j)$-entry $A_{ji}$.

Let $A$ be a matrix.

The transpose of $A$ is the matrix $A^t$ given by
$$(A^t)_{ij} := A_{ji}.$$

The conjugate of $A$ is the matrix $\overline{A}$ given by
$$(\overline{A})_{ij} := \overline{A_{ij}}$$

The conjugate transpose of $A$ is the matrix $\overline{A}^t$ given by $(\overline{A}^t)_{ij} := \overline{A_{ji}}$.

Proof of the theorem If
$$f^*(b_j) = p_{1j}b_1 + p_{2j}b_2 + \cdots + p_{kj}b_k$$
then
$$p_{ij} = \langle f^*(b_j), b_i\rangle = \overline{\langle b_i, f^*(b_j)\rangle}$$
$$= \overline{\langle f(b_i), b_j\rangle}$$
$$= \overline{\langle q_{1i}b_1 + q_{2i}b_2 + \cdots + q_{ki}b_k, b_j\rangle}$$
$$= \overline{q_{ji}}.$$

So $B_{f^*} = (p_{ij})$ and $B_f = (q_{ij})$ and
$$(B_{f^*})_{ij} = p_{ij} = \overline{q_{ji}} = (\overline{B_f}^t)_{ij}.$$

So $B_{f^*} = \overline{B_f}^t$. //

Corollary Let $V$ be a vector space over $\mathbb{C}$ which is finite dimensional and let $\langle , \rangle : V \times V \to \mathbb{C}$ be a positive definite Hermitian form.
Let $f: V \to V$ be a ~~Hermitian~~ linear transformation.
Let $g: V \to V$ be a linear transformation.
Then
(a) $f^*: V \to V$ is a linear transformation and is unique,
(b) $f^* + g^* = (f+g)^*$
(c) $(fg)^* = g^* f^*$.
(d) If $c \in \mathbb{C}$ then $(cf)^* = \bar{c} f^*$
(e) $(f^*)^*$.

Idea of proof: (a) $f^*$ has matrix $\mathcal{B}_{f^*} = \overline{(\mathcal{B}_f)}^t$ as given in the theorem.
~~Next~~ Let $\mathcal{B} = \{b_1, \ldots, b_k\}$ be an orthonormal basis and let

$$A = \mathcal{B}_{f^*} \quad \text{and} \quad C = \mathcal{B}_{g^*}.$$

Then show that
(b') $\overline{(A+C)}^t = \bar{A}^t + \bar{C}^t$
(c') $\overline{(AC)}^t = \bar{C}^t \bar{A}^t$
(d') If $c \in \mathbb{C}$ then $\overline{(cA)}^t = \bar{c} \bar{A}^t$
(e') $\overline{(\bar{A}^t)}^t = A$.