Lecture 17: Group Theory and linear algebra: Spectral Theorem

Let V be a finite dimensional vector space

Let $\langle , \rangle : V \times V \to \mathbb{C}$ be a positive definite Hermitian form.

Let $f: V \to V$ be a linear transformation.

The adjoint to f is $f^*: V \to V$ such that

if $u, w \in V$ then $\langle f(u), w \rangle = \langle u, f^*(w) \rangle$.

The linear transformation f is

- self adjoint, or Hermitian, if f satisfies
  $$f = f^*$$
- an isometry, or Unitary, if f satisfies
  $$f^* f = 1$$
- is normal, if f satisfies
  $$f f^* = f^* f$$

Let A be an $n \times n$ matrix.

The matrix A is

- self adjoint, or Hermitian, if A satisfies
  $$A = \bar{A}^t$$
- an isometry, or unitary, if A satisfies
  $$\bar{A}^t A = 1$$
- normal, if A satisfies
  $$A \bar{A}^t = \bar{A}^t A.$$

Theorem Let V be an inner product space and let $B = \{b_1, b_2, \ldots, b_k\}$ be an orthonormal basis of V. Then

$$C = \{f(b_1), f(b_2), \ldots, f(b_k)\}$$

is an orthonormal basis of V if and only if f is unitary.

Theorem (Spectral Theorem). Let V be an inner product space and let $f: V \to V$ be a normal linear transformation. Let $B = \{b_1, \ldots, b_k\}$ be a ~~orth~~ orthonormal basis of V and let $A = B_f$ be the matrix of f with respect to B. Then, there exists a unitary matrix P such that

$$PAP^{-1} \text{ is diagonal.}$$

Idea of proof:

Show that A and $\bar{A}^t$ have a common eigenvector v_1 (so that $Av_1 = \lambda v_1$ and $\bar{A}^t v_1 = \mu v_1$).

Let $U_1 = \text{span}\{v_1\}$ and write $V = U_1 \oplus U_1^\perp$.

Show that A and $\bar{A}^t$ have a common eigenvector $v_2 \in U_1^\perp$.

Let $U_2 = \text{span}\{v_2\}$ and write $V = U_1 \oplus (U_2 \oplus U_2^\perp)$

Continue. to get $C = \{v_1, v_2, \ldots, v_k\}$. //

<u>Example</u> Let $A=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$, $\zeta=\dfrac{-1+\sqrt{3}i}{2}$

$\zeta^2=\dfrac{-1-\sqrt{3}i}{2}$.

Then

$$\bar{A}^t=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \text{ and } A\bar{A}^t=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\bar{A}^tA.$$

So A is a <u>normal</u> matrix.

Then $v_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector $Av_1=v_1$

If $U_1=\text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ then

$$U_1^\perp=\left\{\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\middle|\ a_1+a_2+a_3=0\right\}.$$

Then $v_2=\begin{pmatrix}1\\ \frac{-1-\sqrt{3}i}{2}\\ \frac{-1+\sqrt{3}i}{2}\end{pmatrix}$ is an eigenvector $Av_2=\zeta v_2$.

$U_2=\text{span}\{v_2\}=\text{span}\left\{\begin{pmatrix}1\\ \frac{-1-\sqrt{3}i}{2}\\ \frac{-1+\sqrt{3}i}{2}\end{pmatrix}\right\}$ is a subspace of $U_1^\perp$

and its complement in $U_1^\perp$ is

$$U_2^\perp=\left\{\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\middle|\ \begin{matrix}a_1+a_2+a_3=0\\ \bar{a}_1+\zeta^2\bar{a}_2+\zeta\bar{a}_3=0\end{matrix}\right\}$$

$=\text{span}\left\{\begin{pmatrix}1\\ \zeta\\ \zeta^2\end{pmatrix}\right\}$ since $\dim(U_2^\perp)=1$ and $1+\zeta^2\cdot\zeta^2+\zeta\cdot\zeta=1+\zeta+\zeta^2=0$.

(Note: $\bar{\zeta}=\zeta^2$ and $\overline{\zeta^2}=\zeta$). Let $v_3=\begin{pmatrix}1\\ \zeta\\ \zeta^2\end{pmatrix}$

With respect to the basis $\{v_1,v_2,v_3\}=B$,

$$B_A=\begin{pmatrix}1&0&0\\0&\zeta&0\\0&0&\zeta^2\end{pmatrix}$$

If $P = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{pmatrix}$ is the change of basis matrix

from $S = \{e_1, e_2, e_3\}$ with $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

to $B = \{v_1', v_2', v_3'\}$ with $v_1' = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2' = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \zeta^2 \\ \zeta \end{pmatrix}$, $v_3' = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \end{pmatrix}$

so that both S and B are orthonormal,

Then

$$P^{-1} = \bar{P}^t = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix}, \text{ since } P \text{ is unitary.}$$

and

$$P A P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix} \text{ is diagonal}$$