Lecture 19. Group Theory and linear algebra, 06.09.2011 (1)

Polar decomposition.

Let $f: V \to V$ be a linear transformation.

Let $\langle\ \rangle: V \times V \to \mathbb{C}$ be a positive definite Hermitian form.

Show that the following are equivalent.

(a) f is self adjoint and all eigenvalues are positive

(b) There exists $g: V \to V$ such that g is self adjoint and $f = g^2$

(c) There exists $h: V \to V$ such that $f = hh^*$

(d) f is self adjoint and $\langle f(v), v\rangle \geq 0$ for all $v \in V$.

Proof To show: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Assume f is self adjoint and all eigenvalues are positive.

To show: There exists $g: V \to V$ such that g is self adjoint and $f = g^2$.

Since f is self adjoint, f is normal.

By the spectral theorem, there exists an orthonormal basis $B = \{b_1, b_2, \ldots, b_k\}$ such that

$$B_f = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_k \end{pmatrix}.$$

Since all eigenvalues of f are positive, $d_1, d_2, \ldots, d_k \in \mathbb{R}_{\geq 0}$.

Let $B_g = \begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_k} \end{pmatrix}$ be the matrix of $g: V \to V$

To show: (1) g is self adjoint

(2) $f = g^2$.

(1) To show: $g = g^*$

$$B_{g^*} = \overline{B_g}^t = \begin{pmatrix} \overline{\sqrt{d_1}} & & 0 \\ & \ddots & \\ 0 & & \overline{\sqrt{d_k}} \end{pmatrix} = \begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_k} \end{pmatrix} = B_g,$$

since $\sqrt{d_1}, \ldots, \sqrt{d_k} \in \mathbb{R}$.

So $g^* = g$.

(2) To show: $f = g^2$

$$B_{g^2} = (B_g)^2 = \begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_k} \end{pmatrix}^2 = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{pmatrix} = B_f$$

So $g^2 = f$.

(b) $\Rightarrow$ (c) Assume there exists $g: V \to V$ such that g is self adjoint and $f = g^2$

To show: There exists $h: V \to V$ such that $f = hh^*$.

Let $h = g$

To show: $f = hh^*$

$hh^* = gg^* = gg = g^2 = f$. ($g = g^*$ since f is self adjoint)

(c) $\Rightarrow$ (d) Assume there exists $h: V \to V$ such that $f = hh^*$.

To show: (1) f is self adjoint

(2) If $v \in V$ then $\langle f(v), v\rangle \geq 0$.

(1) To show: $f = f^*$.

$$f^* = (hh^*)^* = (h^*)^* h^*, \quad \text{since } (ab)^* = b^* a^*,$$
$$= hh^* = f, \quad \text{since } (a^*)^* = a.$$

(2) Assume $v \in V$.

To show: $\langle f(v), v\rangle \in \mathbb{R}_{\geq 0}$.

$$\langle f(v), v\rangle = \langle hh^* v, v\rangle = \langle h^* v, h^* v\rangle \in \mathbb{R}_{\geq 0},$$
since $\langle , \rangle$ is positive definite.

(d) $\Rightarrow$ (a) Assume f is self adjoint and if $v \in V$ then $\langle f(v), v\rangle \in \mathbb{R}_{\geq 0}$.

To show: (1) f is self adjoint

(2) All eigenvalues of f are positive.

(1) To show: f is self adjoint.

By assumption, f is self adjoint.

(2) To show: All eigenvalues of f are positive.

To show: If $\lambda \in \mathbb{C}$ and $v \in V$ and $fv = \lambda v$ then $\lambda \in \mathbb{R}_{\geq 0}$

Assume $\lambda \in \mathbb{C}$ and $v \in V$ and $fv = \lambda v$.

To show: $\lambda \in \mathbb{R}_{\geq 0}$.

We know: $\langle f(v), v \rangle \in \mathbb{R}_{\geq 0}$

So

$$\langle fv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle \in \mathbb{R}_{\geq 0}$$

Since $\langle v, v \rangle \in \mathbb{R}_{\geq 0}$, because $\langle\ ,\ \rangle$ is pos. definite, then $\lambda \in \mathbb{R}_{\geq 0}$ //.

<u>Theorem</u> Let $A \in GL_n(\mathbb{C})$.

Then there exist P, diagonalisable with positive eigenvalues, and U, unitary, such that

$$A = PU.$$

<u>Idea of Proof</u>: Let

P be such that $P^2 = A\bar{A}^t$, and

$U = P^{-1}A$

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