Lecture 20 Group Theory and Linear algebra 07.09.2011

Let G be a group and $g \in G$.

The order of G is $\mathrm{Card}(G)$ and

the order of g is the smallest $k \in \mathbb{Z}_{>0}$ such that

$$g^k = 1.$$

The symmetric group S_n

$S_1 = \{ 1 \}$ with

	1
1	1

$\mathrm{Card}(S_1) = 1$

$S_2 = \{ 11, X \}$ with

	11	X
11	11	X
X	X	11

$\mathrm{Card}(S_2) = 2$

$S_3 = \{ 111, X1, 1X, ⋊, ⋉, ✳ \}$ with

	111	X1	1X	⋊	⋉	✳
111	111	X1	1X	⋊	⋉	✳
X1	X1	111	⋉			
1X	1X	⋊	111			
⋊	⋊					
⋉	⋉					
✳	✳					

since

$X1 \cdot 1X = \begin{matrix} X1 \\ 1X \end{matrix} = ⋉$

$1X \cdot X1 = \begin{matrix} 1X \\ X1 \end{matrix} = ⋊$

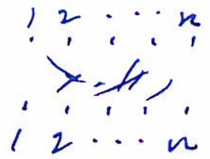
$✳ \cdot ✳ = \begin{matrix} ✳ \\ ✳ \end{matrix} = 111$

$X1 \cdot X1 = \begin{matrix} X1 \\ X1 \end{matrix} = 111$

Then $\mathrm{Card}(S_3) = 6$.

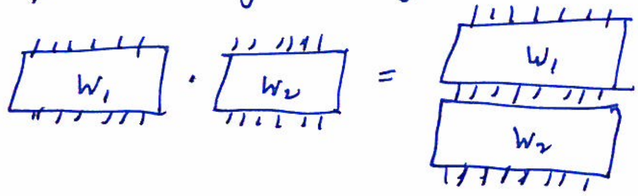
The order of ⋉ is 3 and the order of 1X is 2.

The symmetric group S_n is the set

$S_n = \Big\{$  $\Big|$ each top dot is connected to a unique bottom dot, each bottom dot is connected to some top dot, no two top dots are connected to the same bottom dot $\Big\}$

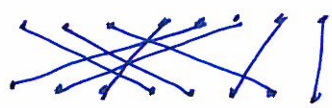
= { bijective function from $1\ 2 \cdots n$ to $1\ 2 \cdots n$ }

with product given by



Different representations of the same permutation

A permutation is an element of S_n

Say $w =$  (diagram notation)

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 7 & 3 & 1 & 2 & 6 & 8 \end{pmatrix}$$ (two line notation)

$$= (1437625)(8)$$ (cycle notation)

$$= \begin{pmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$ (matrix notation)

Let G be a group.

Let S be a subset of G.

The <u>subgroup generated by S</u> is the subgroup $\langle S \rangle \subseteq G$ such that

(a) $S \subseteq \langle S \rangle$

(b) If H is a subgroup of G and $S \subseteq H$ then $\langle S \rangle \subseteq H$.

So $\langle S \rangle$ is the smallest subgroup of G containing S.

<u>Example</u> $G = S_3$, $S = \{ \backslash X \}$.

Then $\langle S \rangle = \{ III, \backslash X, X/ \}$ with

	III	\X	X/
III	III	\X	X/
\X	\X	X/	III
X/	X/	III	\X

<u>Example</u> $G = S_3$, $S = \{ XI \}$

Then $\langle S \rangle = \{ III, XI \}$ with

	III	XI
III	III	XI
XI	XI	III

<u>Note</u>:

$\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ with

+	0	1
0	0	1
1	1	0

$\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ with

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

The function $f : \mathbb{Z}/3\mathbb{Z} \to \{|||, \text{XX}, \text{XX}\}$

$0 \longmapsto |||$

$1 \longmapsto \text{XX}$

$2 \longmapsto \text{XX}$

is an isomorphism.

The subgroups of S_3 are