Lecture 21: Group Theory and linear algebra 09.09.2011

The general linear group with entries in $\mathbb{C}$ is

$$GL_n(\mathbb{C}) = \left\{ \begin{array}{l} n \times n \text{ matrices } g \text{ with entries in } \mathbb{C} \\ \text{such that } g \text{ is invertible} \end{array} \right\}$$

$$= \{ g \in M_n(\mathbb{C}) \mid g^{-1} \in M_n(\mathbb{C}) \}$$

$$= \{ g \in M_n(\mathbb{C}) \mid \det(g) \neq 0 \}$$

So

$$GL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{c} a,b,c,d \in \mathbb{C} \text{ and} \\ ad-bc \neq 0 \end{array} \right\}$$

with product matrix multiplication.

So

$$GL_1(\mathbb{C}) = \{ g \in M_1(\mathbb{C}) \mid \det(g) \neq 0 \}$$

$$= \{ c \in \mathbb{C} \mid c \neq 0 \} = \mathbb{C} - \{0\} = \mathbb{C}^\times$$

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A cyclic group is a group generated by one element.

Examples: $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ is generated by $S = \{1\}$.

$\{|||, \cancel{|||}, \cancel{\cancel{|||}}\}$ is generated by $S = \{\cancel{|||}\}$.

$\{||||, \cancel{||||}, \cancel{\cancel{||||}}, \cancel{\cancel{\cancel{||||}}}\}$ is generated by $S = \{\cancel{||||}\}$

$\{|||||, \cancel{||||}, \cancel{\cancel{||||}}, \cancel{\cancel{\cancel{||||}}}, \cancel{\cancel{\cancel{\cancel{||||}}}}\}$ is generated by $S = \{\cancel{||||}\}$.

$\{1, g, g^2, g^3, g^4\}$ with $g^5 = 1$ is generated by g.

In this last example $g^3 g^4 = g^7 = g^5 g^2 = 1 \cdot g^2 = g^2$.

Another example: $\{1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}\}$ (a subset of $\mathbb{C}$) is a group under multiplication. If

$\zeta = \frac{-1+\sqrt{3}i}{2}$ then $\zeta^2 = \frac{-1-\sqrt{3}i}{2}$ and $\zeta^3 = 1$

so that

$$\left\{1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}\right\} = \{1, \zeta, \zeta^2\} \text{ with } \zeta^3 = 1$$

and this group is generated by $\zeta = \{\zeta\} = \left\{\frac{-1+\sqrt{3}i}{2}\right\}$

The group of n^{th} roots of unity is

$$\mu_n = \{z \in \mathbb{C} \mid z^n = 1\}.$$

The group of 3^{rd} roots of unity is $\left\{1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}\right\} = \mu_3$

Then $\mu_n = \{z \in \mathbb{C} \mid z^n = 1\}$ is a subgroup of $GL_1(\mathbb{C}) = \mathbb{C}^\times$

and

μ_n is generated by $\zeta = e^{2\pi i/n}$

where

$$e^{2\pi i/n} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right).$$

Note: $e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = \frac{-1+\sqrt{3}i}{2}$

Let G and H be groups.

The <u>product</u> of G and H is the set

$$G \times H = \{(g,h) \mid g \in G \text{ and } h \in H\}$$

with

$$(g_1,h_1)(g_2,h_2) = (g_1 g_2, h_1 h_2)$$

<u>Example</u> $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ and

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{(0,0), (0,1), (1,0), (1,1)\}$ with

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

Order of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is 4

Order of $(0,0)$ is 1

Order of $(1,0)$ is 2

Order of $(0,1)$ is 2

Order of $(1,1)$ is 2

<u>Example</u> $\{IIII, XII, IIX, XX\}$ is a subgroup of S_4

	IIII	XII	IIX	XX
IIII	IIII	XII	IIX	XX
XII	XII	IIII	XX	IIX
IIX	IIX	XX	IIII	XII
XX	XX	IIX	XII	IIII

Order of $\{IIII, XII, IIX, XX\}$ is 4

Order of IIII is 1

Order of XII is 2

Order of IIX is 2

Order of XX is 2

and the function

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \{IIII, XII, IIX, XX\}$$

$$\begin{aligned}
(0,0) &\longmapsto IIII \\
(1,0) &\longmapsto XII \\
(0,1) &\longmapsto IIX \\
(1,1) &\longmapsto XX
\end{aligned}$$

is an isomorphism.

<u>Subgroups of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:</u>

$$\begin{tikzcd}
 & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{(0,0),(1,0)\} \arrow[dr, dash] & \{(0,0),(0,1)\} \arrow[d, dash] & \{(0,0),(1,1)\} \arrow[dl, dash] \\
 & \{(0,0)\} &
\end{tikzcd}$$

<u>Subgroups of $\{IIII, XII, IIX, XX\}$:</u>

$$\begin{tikzcd}
 & \{IIII, XII, IIX, XX\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{IIII, XII\} \arrow[dr, dash] & \{IIII, IIX\} \arrow[d, dash] & \{IIII, XX\} \arrow[dl, dash] \\
 & \{IIII\} &
\end{tikzcd}$$