Lecture 22, Group Theory and Linear algebra 13.09.2011 ①

Let G be a group.

A subgroup of G is a subset $H \subseteq G$ such that

(a) If $h_1, h_2 \in H$ then $h_1 h_2 \in H$,

(b) $1 \in H$

(c) If $h \in H$ then $h^{-1} \in H$.

Let H be a subgroup of G.

A coset of H in G is a subset

$$gH = \{ gh \mid h \in H \}$$

with $g \in G$.

Example $G = \{111, X1, 1X, ⋇, \X, X/\} = S_3$

$H = \{111, X1\}$.

Then $111 \cdot H = \{111, X1\}$

$X1 \cdot H = \{X1, 111\}$

$1X \cdot H = \{1X, \X\}$

$⋇ \cdot H = \{⋇, X/\}$ are the cosets of H in S_3.

$\X \cdot H = \{\X, 1X\}$

$X/ \cdot H = \{X/, ⋇\}$

There are really only 3 cosets here since

$111 \cdot H = X1 \cdot H = \{111, X1\}$

$1X \cdot H = \X \cdot H = \{1X, \X\}$

$⋇ \cdot H = X/ \cdot H = \{⋇, X/\}$

$G/H = \{gH \mid g \in G\}$ is the set of cosets of H in G

In our example

$$S_3/H = \{ III \cdot H,\ IX \cdot H,\ \divideontimes \cdot H \}$$

$$= \{ \{III, XI\},\ \{IX, \text{XX}\},\ \{\divideontimes, \text{XX}\} \}.$$

<u>Proposition</u> Let G be a group and let H be a subgroup of G. Let $g \in G$.

(a) $\mathrm{Card}(gH) = \mathrm{Card}(H)$

(b) G/H is a partition of G.

A <u>partition</u> of a set S is a collection $\mathcal{S}$ of subsets of S such that

(a) The union of the sets in $\mathcal{S}$ is S,

(b) If $U_1, U_2 \in \mathcal{S}$ then $U_1 = U_2$ or $U_1 \cap U_2 = \phi$.

<u>Corollary</u> Let G be a group and let H be a subgroup of G. Then

$$\mathrm{Card}(G) = \mathrm{Card}(G/H)\,\mathrm{Card}(H)$$

<u>Proof of the proposition</u>

(a) To show: $\mathrm{Card}(gH) = \mathrm{Card}(H)$

To show: There exists a bijective function $f: H \to gH$.

Let
$$\begin{array}{rl} f: H & \to gH \\ h & \mapsto gh. \end{array}$$

To show: f is bijective.

To show: There exists a function $\varphi: gH \to H$ such that $\varphi \circ f = id_H$ and $f \circ \varphi = id_{gH}$.

Let
$$\begin{array}{rl} \varphi: gH & \to H \\ \cancel{gh} & \cancel{\mapsto h}, \\ x & \mapsto g^{-1}x \end{array}$$ so that $\varphi(x) = g^{-1}x$

To show: (aa) $\varphi \circ f = id_H$

(ab) $f \circ \varphi = id_{gH}$.

(aa) To show: If $h \in H$ then $(\varphi \circ f)(h) = id_H(h)$.

Assume $h \in H$.

To show: $(\varphi \circ f)(h) = id_H(h)$.

$$(\varphi \circ f)(h) = \varphi(f(h)) = \varphi(gh) = g^{-1}gh = h = id_H(h).$$

(ab) To show: If $x \in gH$ then $(f \circ \varphi)(x) = id_{gH}(x)$.

Assume $x \in gH$

To show: $(f \circ \varphi)(x) = id_{gH}(x)$.

$(f\circ\varphi)(x) = f(\varphi(x)) = f(g^{-1}x) = gg^{-1}x = x = id_{gH}(x).$

(b) To show: G/H is a partition of G.

To show: (ba) $\bigcup_{g\in G} gH = G$

(bb) If $g_1, g_2 \in G$ then $g_1H = g_2H$ or $g_1H \cap g_2H = \emptyset$

(ba) To show: If $x \in G$ then there exists $g \in G$ such that $x \in gH$.

Assume $x \in G$

To show: There exists $g \in G$ such that $x \in gH$.

Let $g = x$.

To show: $x \in gH$.

~~To show~~ $x = x \cdot 1 \in xH$, since $1 \in H$.

(bb) Assume $g_1, g_2 \in G$ and $g_1H \cap g_2H \neq \emptyset$

To show: $g_1H = g_2H$.

~~Assume~~ Since $g_1H \cap g_2H \neq \emptyset$ there exists $x \in g_1H \cap g_2H$.

Let $h_1, h_2 \in H$ be such that $x = g_1h_1$ and $x = g_2h_2$.

To show: $g_1H = g_2H$.

To show: (bba) $g_1H \subseteq g_2H$

(bbb) $g_2H \subseteq g_1H$.

(bba) To show: If $y \in g_1H$ then $y \in g_2H$.

Assume $y \in g_1H$.

Then there exists $h \in H$ such that $y = g_1h$.

To show $y \in g_2H$.

$$y = g_1h = g_1h_1h_1^{-1}h = xh_1^{-1}h = g_2h_2h_1^{-1}h \in g_2H,$$

since H is a subgroup.

(bbb) To show: If $z \in g_2H$ then $z \in g_1H$.

Assume $z \in g_2H$.

Then there exists $h' \in H$ such that $z = g_2h'$

To show: $z \in g_1H$.

$$z = g_2h' = g_2h_2h_2^{-1}h' = xh_2^{-1}h' = g_1h_1h_2^{-1}h' \in g_1H,$$

since H is a subgroup.

So $g_1H = g_2H$.

So G/H is a partition of G //.