Lecture 24: Group theory and linear algebra. 16.09.2011

~~Let~~ Group homomorphisms are for comparing groups.

Let G and H be groups.

A group homomorphism from G to H is a function $f: G \to H$ such that

(a) If $g_1, g_2 \in G$ then $f(g_1 g_2) = f(g_1) f(g_2)$.

Let $f: G \to H$ be a group homomorphism.

The kernel of f is

$$\ker f = \{ g \in G \mid f(g) = 1 \}.$$

The image of f is

$$\operatorname{im} f = \{ f(g) \mid g \in G \}.$$

Lemma Let $f: G \to H$ be a group homomorphism.

Then $\ker f$ is a normal subgroup of G.

Proof of crucial point:

To show: If $g \in G$ and $k \in \ker f$ then $gkg^{-1} \in \ker f$.

Assume $g \in G$ and $k \in \ker f$.

To show: $gkg^{-1} \in \ker f$

To show: $f(gkg^{-1}) = 1$.

Since $k \in \ker f$, $f(k) = 1$.

Since $f(gg^{-1}) = f(1) = 1$ then $f(g^{-1}) = f(g)^{-1}$

So $f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)f(k)f(g)^{-1}$

$= f(g)\cdot 1\cdot f(g)^{-1} = 1.$ //

<u>Proposition</u> Let $f: G \to H$ be a group homomorphism.
Let $K = \ker f$

(a) The function $f': G \to \mathrm{im} f$
$g \mapsto f(g)$ is a
surjective group homomorphism.

(b) The function $\hat{f}: G/K \to H$
$gK \mapsto f(g)$ is a well
defined injective group homomorphism.

(c) The function $\hat{f}': G/K \to \mathrm{im} f$
$gK \mapsto f(g)$ is a well
defined bijective group homomorphism.

<u>Proof</u> in order of most crucial points to least crucial.

(ba) To show: The function $\hat{f}: G/K \to H$ given
by $\hat{f}(gK) = f(g)$ is well defined. (i.e. <u>is</u> a function).

To show: If $g_1K = g_2K$ then $\hat{f}(g_1K) = \hat{f}(g_2K)$

Assume $g_1K = g_2K$.

Then $g_1 \in g_2K$

So there exists $k \in K$ such that $g_1 = g_2k$

To show: $\hat{f}(g_1K) = \hat{f}(g_2K)$

To show: $f(g_1) = f(g_2)$

$f(g_1) = f(g_2k) = f(g_2)f(k) = f(g_2)\cdot 1 = f(g_2)$ //

(bb) To show: The function $\hat{f}: G/K \to H$ given by $\hat{f}(gK) = f(g)$ is injective.

To show: If $g_1K, g_2K \in G/K$ and $\hat{f}(g_1K) = \hat{f}(g_2K)$ then $g_1K = g_2K$.

Assume $g_1K, g_2K \in G/K$ and $\hat{f}(g_1K) = \hat{f}(g_2K)$

Then $f(g_1) = f(g_2)$.

To show: $g_1K = g_2K$.

To show: $g_1 \in g_2K$, since G/K partitions G.

To show: There exists $k \in K$ such that $g_1 = g_2k$

To show: $k = g_2^{-1}g_1$ is an element of $K = \ker f$.

To show $f(g_2^{-1}g_1) = 1$.

$f(g_2^{-1}g_1) = f(g_2^{-1})f(g_1) = f(g_2)^{-1}f(g_1) = f(g_2)^{-1}f(g_2) = 1$ //

⋮

<u>Example</u> $G = GL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{C} \text{ and} \\ ad-bc \neq 0 \end{matrix} \right\}.$

Then
$$f : GL_2(\mathbb{C}) \longrightarrow GL_1(\mathbb{C})$$
$$A \longmapsto \det(A)$$
is a group homomorphism

because $\det(A_1 A_2) = \det(A_1)\det(A_2)$.

Then
$$\ker f = \{ A \in GL_2(\mathbb{C}) \mid f(A) = 1 \}$$
$$= \{ A \in GL_2(\mathbb{C}) \mid \det(A) = 1 \}$$
$$= SL_2(\mathbb{C})$$

Since $\operatorname{im} f = \mathbb{C}^\times$ (because $\det\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a$)

then
$$G/\ker f \xrightarrow{\sim} \mathbb{C}^\times$$

So $\dfrac{GL_2(\mathbb{C})}{SL_2(\mathbb{C})} \xrightarrow{\sim} \mathbb{C}^\times$ and $SL_2(\mathbb{C})$ is a normal subgroup of $GL_2(\mathbb{C})$.