Lecture 25: Group Theory and Linear algebra, Group Actions

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The dihedral group D_n is the set

$$D_n = \{1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y\}$$

with operation given by

$$(x^i y^j)(x^k y^l) = x^{(i-k \bmod n)} y^{(j+l \bmod 2)}$$

so that, in particular

$$y^2 = 1, \quad x^n = 1 \text{ and } \quad yx = x^{-1}y.$$

A G-set S, or

An action of a group G on a set S is a function

$$\begin{aligned} G \times S &\to S \\ (g, s) &\mapsto gs \end{aligned} \quad \text{such that}$$

(a) If $g_1, g_2 \in G$ and $s \in S$ then $g_1(g_2 s) = (g_1 g_2)s$,

(b) If $s \in S$ then $1 \cdot s = s$.

Let $s \in S$.

The stabilizer of s is

$$G_s = \{g \in G \mid gs = s\}$$

The orbit of s is

$$Gs = \{gs \mid g \in G\}.$$

<u>Proposition</u> Let G be a group and let S be a G-set.

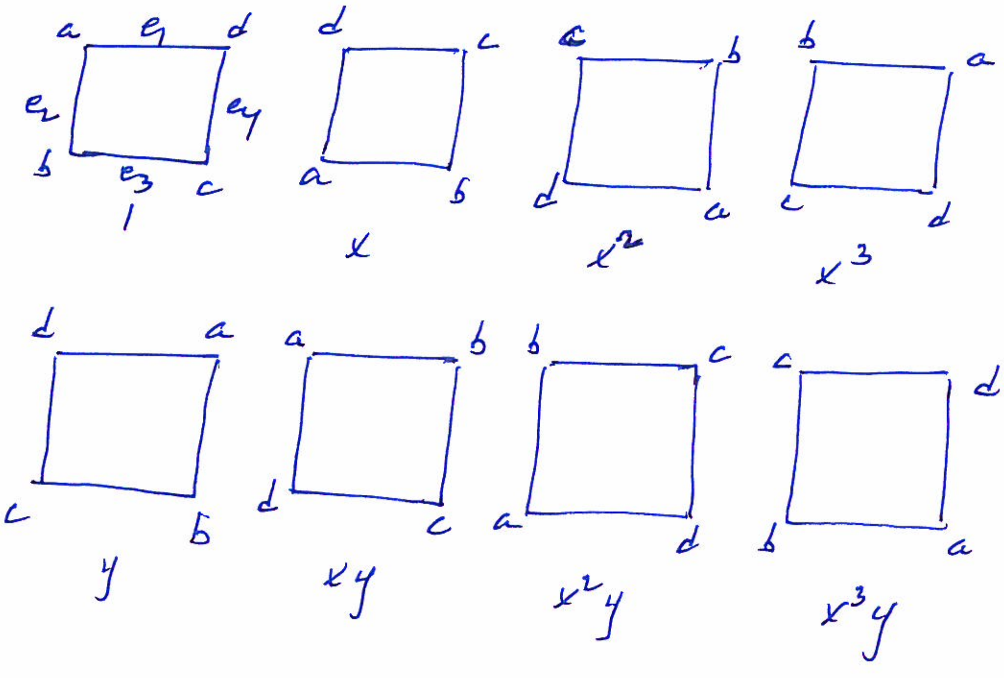
(a) The orbits of G acting on S partitions S

(b) Let $s \in S$. Then G_s is a subgroup of G,

$$\varphi : G/G_s \to Gs$$
$$gG_s \mapsto gs$$

is a well defined function and φ is a bijection.

<u>Example</u> "D_4 acting on a square".



D_4 acts on the vertices $V = \{a, b, c, d\}$:

$$xa = b, \qquad xya = a$$

D_4 acts on the edges $E = \{e_1, e_2, e_3, e_4\}$

$$x^2 e_1 = e_3, \qquad y e_1 = e_1.$$

<u>Example</u> "G acts on itself by conjugation"

$$G \times G \longrightarrow G$$
$$(g, s) \longmapsto gsg^{-1} \qquad \text{or} \qquad g \cdot s = gsg^{-1},$$

where $g \in G$ and $s \in G$.

Let $s \in G$. The <u>centraliser</u> of s in G is the stabiliser of s,

$$Z_G(s) = \mathrm{Stab}(s) = \{ g \in G \mid gsg^{-1} = s \}$$

The <u>conjugacy class</u> of s in G is the orbit of s,

$$\mathcal{C}_s = \{ gsg^{-1} \mid g \in G \}$$

The <u>centre</u> of G is

$$Z(G) = \{ s \in G \mid gs = sg \text{ for all } g \in G \}$$

HW: Show that $s \in Z(G)$ if and only if $Z_G(s) = G$.

HW: Show that $s \in Z(G)$ if and only if $\mathcal{C}_s = \{s\}$.

<u>Example</u> D_4 acts on itself by conjugation

$x \circlearrowright 1 \circlearrowleft y$ and $x \circlearrowright x^2 \circlearrowleft y$

$x \circlearrowright x \underset{y}{\overset{y}{\rightleftarrows}} x^3 \circlearrowleft x$ since $yxy^{-1} = x^3yy^{-1} = x^3$

$yx^2y^{-1} = x^2yy^{-1} = x^2$

$yx^3y^{-1} = xyy^{-1} = x$

$y \circlearrowright y \underset{x}{\overset{x}{\rightleftarrows}} x^2y \circlearrowleft y$ since $xyx^{-1} = xyx^3 = xxy = x^2y$

$xy \underset{x,y}{\overset{y,x}{\rightleftarrows}} x^3y$

So the conjugacy classes are

$\{1\}, \{x^2\}, \{x, x^3\}, \{y, x^2y\}, \{xy, x^3y\}$

and the centralizers of elements in D_4 are

$Z_{D_4}(1) = Stab(1) = D_4$

$Z_{D_4}(x) = Stab(x) = \{1, x, x^2, x^3\}$

$Z_{D_4}(x^2) = Stab(x^2) = D_4$

$Z_{D_4}(x^3) = Stab(x^3) = \{1, x, x^2, x^3\}$

$Z_{D_4}(y) = Stab(y) = \{1, x^2, y, x^2y\}$

$Z_{D_4}(xy) = Stab(xy) = \{1, x^2, xy, x^3y\}$

$Z_{D_4}(x^2y) = Stab(x^2y) = \{1, x^2, y, x^2y\}$

$Z_{D_4}(x^3y) = Stab(x^3y) = \{1, x^2, xy, x^3y\}$