Lecture 26, Group Theory and linear algebra, 5 October 2011. 1

p-groups

Let p be a prime in $\mathbb{Z}_{>0}$.

A *p-group* is a group G such that there exists $a \in \mathbb{Z}_{>0}$ with $\mathrm{Card}(G) = p^a$.

Proposition Let G be a p-group, $\mathrm{Card}(G) = p^a$.

(a) G contains an element of order p.

(b) $Z(G) \neq \{1\}$.

Proof (a) To show: There exists $y \in G$ with $\mathrm{order}(y) = 1$.

Let $x \in G$ with $x \neq 1$.

Then $\mathrm{order}(x)$ divides $\mathrm{Card}(G)$ and $\mathrm{order}(x) \neq 1$.

So $\mathrm{order}(x) = p^b$ with $0 < b \leq a$.

Let $y = x^{p^{b-1}}$.

Then $y \neq 1$ and

$$y^p = (x^{p^{b-1}})^p = x^{p^b} = 1.$$

So $\mathrm{order}(y) = p$. //

(b) To show: $Z(G) \neq \{1\}$.

We know that $Z(G)$ is the union of the conjugacy classes of size 1.

We know that, if $\mathcal{C}_s$ is a conjugacy class in G then

$$\mathrm{Card}(\mathcal{C}_s) = \mathrm{Card}(G/\mathrm{Stab}(s)) \text{ divides } \mathrm{Card}(G) = p^a.$$

So, either $\mathrm{Card}(\mathcal{C}_s) = 1$ or $\mathrm{Card}(\mathcal{C}_s)$ is divisible by p.

Then

$$p^a = \mathrm{Card}(G) = \underbrace{1+1+\cdots+1}_{\substack{\text{number of}\\ \text{conj. classes of}\\ \text{size } 1}} + \sum_{\substack{\text{conj. classes } \mathcal{C}_s\\ \text{with}\\ \mathrm{Card}(\mathcal{C}_s)>1}} \mathrm{Card}(\mathcal{C}_s).$$

So

$$\begin{pmatrix}\text{number of conjugacy}\\ \text{classes of size } 1\end{pmatrix} \text{ is divisible by } p.$$

So $\mathrm{Card}(Z(G))$ is divisible by p.

So $Z(G) \neq \{1\}$. //.

<u>Proposition</u> Let G be a group with $\mathrm{Card}(G) = p^2$. Then G is abelian.

<u>Proof</u> To show: $Z = Z(G)$ is all of G.

We know, from (b) of the last Proposition, that $Z \neq \{1\}$

We know, $\mathrm{Card}(Z)$ divides $\mathrm{Card}(G) = p^2$.

So $\mathrm{Card}(Z) = p$ or $\mathrm{Card}(Z) = p^2$.

Case 1: $\mathrm{Card}(Z) = p$.

Let $x \in G$ with $x \notin Z$.

Then xZ generates

$$G/Z = \{Z, xZ, x^2Z, \ldots, x^{p-1}Z\}$$

(Z is a normal subgroup of G and $G/Z \cong \mathbb{Z}/p\mathbb{Z}$).

Let $g \in G$. Then there exists $0 \le k < p$ and $z \in Z$ with

$$g = x^k z.$$

So

$$gx = x^k z \cancel{x} = x^k x z, \qquad \text{since } z \in Z$$
$$= x^{k+1} z = x(x^k z) = xg.$$

So $x \in Z$.

This is a contradiction to $x \notin Z$.

So $\mathrm{Card}(Z) \neq p$.

Case 2 $\mathrm{Card}(Z) = p^2$.

Since $\mathrm{Card}(G) = p^2$, then $Z = G$.

So G is abelian. //

About conjugacy classes, normal subgroups and centres

(1) Let N be a normal subgroup of G.

Then N is a union of conjugacy classes of G.

Proof To show: If $n \in N$ then $\mathcal{C}_n \subseteq N$.

To show: If $n \in N$ and $g \in G$ then $gng^{-1} \in N$.

This is true since N is normal.

(2) Let G be a group and $Z = Z(G)$.

Then Z is a normal subgroup of G.

and

if $z \in Z$ then $\mathcal{C}_z = \{z\}$.

Proof To show: If $z \in Z$ then $\mathcal{C}_z = \{z\}$.

Assume $z \in Z$.

To show: $\mathcal{C}_z = \{z\}$.

$$\mathcal{C}_z = \{ gzg^{-1} \mid g \in G\} = \{ gg^{-1}z \mid g \in G\}$$
$$= \{z\} //$$

If $\mathcal{C}_z = \{z\}$ then $z \in Z$.

Proof Assume $z \in G$ and $\mathcal{C}_z = \{z\}$.

To show: $z \in Z$.

To show: If $g \in G$ then $gz = zg$.

Assume $g \in G$.

Then $gz = (gzg^{-1})g = zg$, since $gzg^{-1} \in \mathcal{C}_z = \{z\}$. //