Lecture 27: Counting orbits and Stabilizers, Group Theory and Linear Algebra 07.10.2011. ①

Let G be a group and let S be a G-set.

Proposition A

(a) The orbits partition S.

(b) If $s \in S$ and $H = \mathrm{Stab}(s)$ then

$$\begin{aligned} \varphi : G/H &\longrightarrow Gs \\ gH &\longmapsto gs \end{aligned} \quad \text{is a function}$$

and φ is a bijection.

Let G be a group and let N be a subgroup of G.

Proposition B

(a) The cosets in G/N partition G

(b) All cosets have the same size.

Idea of proof of (b):

Let $g \in G$.

To show: $\begin{aligned} \varphi : gN &\longrightarrow N \\ gn &\longmapsto n \end{aligned}$ is a function and

φ is a bijection.

Let $\sim$ be an equivalence relation on a set S.

Proposition C

The equivalence classes partition S.

Proof of Proposition A

(a) To show: The orbits partition S.

To show: (aa) $\bigcup_{s\in S} Gs = S$

(ab) If $s_1, s_2 \in S$ and $Gs_1 \cap Gs_2 \neq \phi$ then $Gs_1 = Gs_2$.

(aa) To show: (aaa) $\bigcup_{s\in S} Gs \subseteq S$

(aab) $S \subseteq \bigcup_{s\in S} Gs$.

(aaa) Since $Gs \subseteq S$ then $\bigcup_{s\in S} Gs \subseteq S$.

(aab) To show: If $a \in S$ then $a \in \bigcup_{s\in S} Gs$.

Since $a \in S$, and $a \in Ga$,

then $a \in \bigcup_{s\in S} Gs$.

So $S \subseteq \bigcup_{s\in S} Gs$.

(ab) Assume $s_1, s_2 \in S$ and $Gs_1 \cap Gs_2 \neq \phi$

To show: $Gs_1 = Gs_2$.

Since $Gs_1 \cap Gs_2 \neq \phi$ there exists $t \in Gs_1 \cap Gs_2$.

So there exist $g_1, g_2 \in G$ such that

$g_1 s_1 = t = g_2 s_2$.

So $s_1 = g_1^{-1} g_2 s_2$ and $s_2 = g_2^{-1} g_1 s_1$

To show: (aba) $Gs_1 \subseteq Gs_2$

(abb) $Gs_2 \subseteq Gs_1$

(aba) To show: If $\ell \in Gs_1$, then $\ell \in Gs_2$.

Assume $\ell \in Gs_1$

Then there exists $h \in G$ such that $\ell = hs_1$

So $\ell = hs_1 = hg_1^{-1}g_2 s_2 \in Gs_2$, since $hg_1^{-1}g_2 \in G$.

So $Gs_1 \subseteq Gs_2$.

(abb) To show: If $m \in Gs_2$ then $m \in Gs_1$

Assume $m \in Gs_2$

Then there exists $k \in G$ such that $m = ks_2$.

So $m = ks_2 = kg_2^{-1}g_1 s_1 \in Gs_1$, since $kg_2^{-1}g_1 \in G$.

So $Gs_2 \subseteq Gs_1$

So $Gs_1 = Gs_2$

So the orbits partition G. //.

(b) To show: (ba) $\varphi : G/H \to Gs$, $gH \mapsto gs$ is a function

(bb) φ is a bijection.

(ba) To show: If $g_1H, g_2H \in G/H$ and $g_1H = g_2H$ then $\varphi(g_1H) = \varphi(g_2H)$.

Assume $g_1, g_2 \in G$ and $g_1 H = g_2 H$.

Then $g_1 \in g_2 H$.

So there exists $h \in H$ with $g_1 = g_2 h$.

To show: $\varphi(g_1 H) = \varphi(g_2 H)$.

To show: $g_1 s = g_2 s$.

$g_1 s = g_2 h s = g_2 s$, since $h \in Stab(s)$.

(bb) To show: φ is a bijection.

To show: $\psi : Gs \to G/H$, $t \mapsto gH$ where $t = gs$, $g \in G$ such that, is an inverse function to φ.

To show: (bba) If $g_1, g_2 \in G$ and $g_1 s = g_2 s$ then $\psi(g_1 s) = \psi(g_2 s)$.

(bbb) ~~To s~~ $\varphi \circ \psi = id_{G/H}$ and $\psi \circ \varphi = id_{Gs}$.

(bba) Assume $g_1, g_2 \in G$ and $g_1 s = g_2 s$.

Then $g_1^{-1} g_2 s = s$, so that $g_1^{-1} g_2 \in Stab(s)$.

To show: $\psi(g_1 s) = \psi(g_2 s)$.

To show: $g_1 H = g_2 H$.

To show (bbaa) $g_1 H \subseteq g_2 H$

(bbab) $g_2 H \subseteq g_1 H$.

(bbaa) To show: If $x \in g_1 H$ then $x \in g_2 H$.

Assume $x \in g_1 H$.

Then there exists $h \in H$ such that $x = g_1 h$.

To show: $x \in g_2 h$.

$x = g_1 h = g_2 g_2^{-1} g_1 h \in g_2 H$, since $g_2^{-1} g_1 \in \mathrm{Stab}(s) = H$.

So $g_1 H \subseteq g_2 H$.

(bbab) To show: If $y \in g_2 H$ then $y \in g_1 H$.

Assume $y \in g_2 H$.

Then there exists $k \in H$ such that $y = g_2 k$.

So $y = g_2 k = g_1 g_1^{-1} g_2 k \in g_1 H$, since $g_1^{-1} g_2 \in \mathrm{Stab}(s) = H$.

So $g_2 H \subseteq g_1 H$.

So $g_2 H = g_1 H$.

(bbb) To show: $\varphi \circ \psi = id_{G/H}$ and $\psi \circ \varphi = id_{Gs}$.

If $g \in G$ then

$(\varphi \circ \psi)(gH) = \varphi(\psi(gH)) = \varphi(gs) = gH$ and

$(\psi \circ \varphi)(gs) = \psi(\varphi(gs)) = \psi(gH) = gs$.

So $\varphi \circ \psi = id_{G/H}$ and $\psi \circ \varphi = id_{Gs}$. //