Lecture 28 The affine orthogonal group and isometries

Group Theory and Linear algebra 11.10.2011.

The affine orthogonal group is

$$AO_n(\mathbb{R}) = \left\{ \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array} \right) \middle| \begin{array}{l} g \in O_n(\mathbb{R}) \\ \mu \in \mathbb{R}^n \end{array} \right\}$$

The orthogonal group is

$$O_n(\mathbb{R}) = \{ g \in M_{n\times n}(\mathbb{R}) \mid gg^t = 1 \}.$$

The special orthogonal group is

$$SO_n(\mathbb{R}) = \{ g \in O_n(\mathbb{R}) \mid \det(g) = 1 \}$$

and $O_n(\mathbb{R}) / SO_n(\mathbb{R}) = \{ N, rN \}$ where $N = SO_n(\mathbb{R})$ and $r = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$,

since

$$\begin{aligned} \det : O_n(\mathbb{R}) &\longrightarrow \{\pm 1\} \\ g &\longmapsto \det(g) \end{aligned} \quad \text{and} \quad \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$$

A rotation is an element of $N = SO_n(\mathbb{R})$ and a reflection is an element of rN, where $N = SO_n(\mathbb{R})$ and $r = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$

For $\mu \in \mathbb{R}^n$ and $g \in O_n(\mathbb{R})$ let

$$X^\mu = \left(\begin{array}{c|c} 1 & \mu \\ \hline 0 & 1 \end{array}\right) \quad \text{and} \quad g = \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right)$$

Then

$$g X^\mu g^{-1} = X^{g\mu} \quad \text{and} \quad X^\mu X^\nu = X^{\mu+\nu}$$

since

$$g X^\mu = \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} 1 & \mu \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} g & g\mu \\ \hline 0 & 1 \end{array}\right) \quad \text{and}$$

$$X^{g\mu} g = \left(\begin{array}{c|c} 1 & g\mu \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} g & g\mu \\ \hline 0 & 1 \end{array}\right).$$

Let

$$\mathbb{E}^n = \left\{ \left(\begin{array}{c} x \\ \hline 1 \end{array}\right) \;\middle|\; x \in \mathbb{R}^n \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{pmatrix} \;\middle|\; x_1, x_2, \ldots, x_n \in \mathbb{R} \right\}$$

The group $AO_n(\mathbb{R})$ acts on $\mathbb{E}^n$ by

$$g\left(\begin{array}{c} x \\ \hline 1 \end{array}\right) = \left(\begin{array}{c} gx \\ \hline 1 \end{array}\right) \quad \text{and} \quad X^\mu \left(\begin{array}{c} x \\ \hline 1 \end{array}\right) = \left(\begin{array}{c} \mu + x \\ \hline 1 \end{array}\right)$$

Note that, if $\mu \neq 0$ then

$$\begin{aligned} t_\mu : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto \mu + x \end{aligned}$$

is <u>not</u> a linear transformation, in particular $t_\mu(0) \neq 0$.

Let $d: \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}_{\geq 0}$ be the metric on $\mathbb{E}^n$ given by

$$d(x,y) = |x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

for $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Let $\langle\,,\rangle: \mathbb{E}^n \times \mathbb{E}^n \to \mathbb{R}$ be the positive definite bilinear form given by

$$\langle x,y \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

for $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Note that

$$d(x,y) = \sqrt{\langle x-y, x-y\rangle} \quad \text{and} \qquad (a)$$

$$\begin{aligned} \langle x,y\rangle &= \tfrac{1}{4}\big(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle\big) \\ &= \tfrac{1}{4}\big(d(x,-y)^2 - d(x,y)^2\big) \end{aligned} \qquad (b)$$

An <u>isometry</u> of $\mathbb{E}^n$ is a function $f: \mathbb{E}^n \to \mathbb{E}^n$ such that

if $x, y \in \mathbb{E}^n$ then $d(fx, fy) = d(x,y)$.

H.W: Use (a) and (b) to show that if $f: \mathbb{E}^n \to \mathbb{E}^n$ is an isometry then ~~if and only if~~ f satisfies
if $x, y \in \mathbb{E}^n$ then $\langle fx, fy\rangle = \langle x, y\rangle$.

The group of isometries of $\mathbb{E}^n$ is

$$\mathrm{Isom}(\mathbb{E}^n) = \{ f : \mathbb{E}^n \to \mathbb{E}^n \mid f \text{ is an isometry} \}$$

with operation given by composition of functions.

Theorem Define

$$\Phi : AO_n(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathbb{E}^n)$$
$$y \longmapsto f_y$$

where $f_y : \mathbb{E}^n \to \mathbb{E}^n$ is given by

$$f_y\left(\left(\begin{array}{c} x \\ \hline 1 \end{array}\right)\right) = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c} x \\ \hline 1 \end{array}\right) \quad \text{if } y = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right).$$

Then Φ is a group isomorphism.

Let $\mu \in \mathbb{R}^n$. Translation by μ is the function

$$t_\mu : \mathbb{E}^n \longrightarrow \mathbb{E}^n$$
$$\left(\begin{array}{c} x \\ \hline 1 \end{array}\right) \longmapsto \left(\begin{array}{c} \mu + x \\ \hline 1 \end{array}\right)$$

Note that t_μ is an isometry since

$$d(t_\mu x, t_\mu y) = \sqrt{\langle (\mu + x) - (\mu + y), (\mu + x) - (\mu + y) \rangle}$$
$$= \sqrt{\langle x - y, x - y \rangle} = d(x, y).$$