Lecture 29 Group Theory and Linear Algebra 12.10.2011

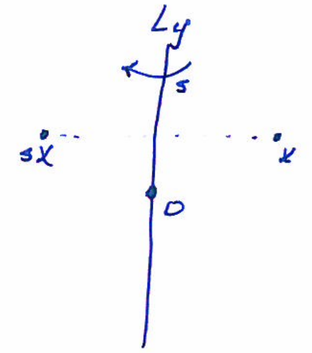
Isometries in $\mathbb{E}^2$

Let

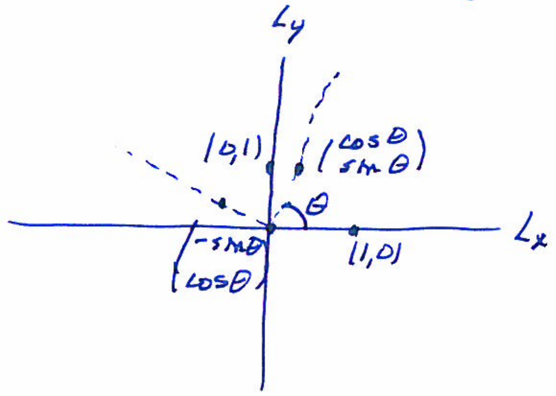
$$s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and let
$$t_\gamma : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$x \longmapsto \gamma + x, \quad \text{for } \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \text{ in } \mathbb{R}^2.$$

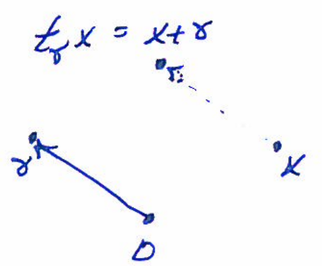
s is reflection in the y-axis L_y



r_θ is rotation in an angle θ about 0



t_γ is translation by γ



$$SO_2(\mathbb{R}) = \{ g \in M_{2\times 2}(\mathbb{R}) \mid gg^t = 1, \det(g) = 1 \}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{R},\ ad-bc=1 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{R},\ a^2+b^2=1,\ ac+bd=0 \\ ca+db=0,\ c^2+d^2=1,\ ad-bc=1 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{ll} a,b,c,d \in \mathbb{R}, & b=-c \\ a^2+b^2=1, & a=d \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R},\ a^2+b^2=1 \right\}$$

$$= \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;\middle|\; 0 \le \theta < 2\pi \right\}$$

$$= \{ r_\theta \mid 0 \le \theta < 2\pi \}.$$

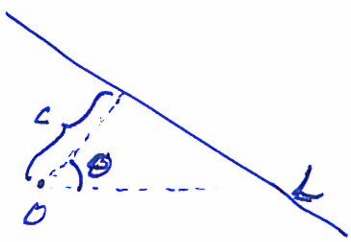
So $SO_2(\mathbb{R})$ is the group of rotations about 0.

(1) Let L be a line in $\mathbb{R}^2$. Then there exist

$c \in \mathbb{R}$ and $0 \le \theta < \pi$

such that

$$L = r_\theta \, t_{\binom{c}{0}} \, L_y$$



The <u>reflection in the line L</u> is

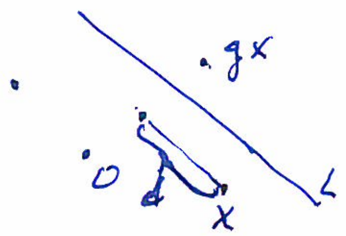
$$s_L = r_\theta \, t_{\binom{c}{0}} \, s \, t_{\binom{-c}{0}} \, r_{-\theta}$$

2. Let $p \in \mathbb{R}^2$ and $\theta \in [0, 2\pi)$

Then rotation by θ around p is

$$r_{\theta,p} = t_p r_\theta t_{-p}.$$

3. The d-glide reflection in the line L is translate by a distance d in a line parallel to L and then reflect in L.



Isometries

Let $\mathbb{E}^2 = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$ with

$$d(p,q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \text{if } p = \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$

An isometry of $\mathbb{E}^2$ is a function $f : \mathbb{E}^2 \to \mathbb{E}^2$ such that

$$d(fp, fq) = d(p,q).$$

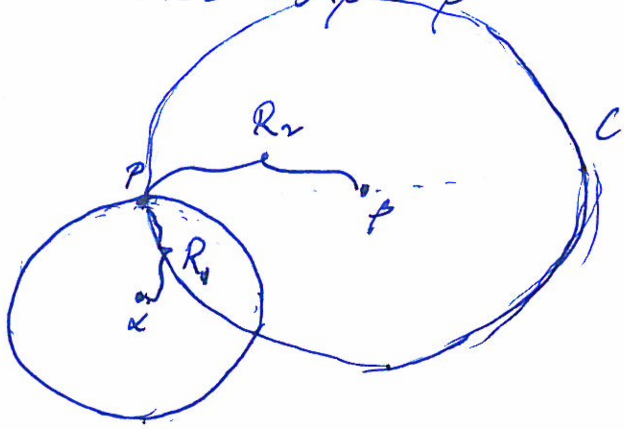
Note that

- a rotation fixes one point
- a reflection fixes a line
- a translation fixes no point.

Let $f : \mathbb{E}^2 \to \mathbb{E}^2$ be an isometry.

Suppose α, β are fixed points of f,

$f\alpha = \alpha$ and $f\beta = \beta$



Let $p \in \mathbb{E}^2$.

Since $d(\alpha, p) = d(f\alpha, fp) = d(\alpha, fp)$,

fp must lie on the circle C_1 of radius $R_1 = d(\alpha, p)$ centred at α.

Since $d(\beta, p) = d(f\beta, fp) = d(\beta, fp)$

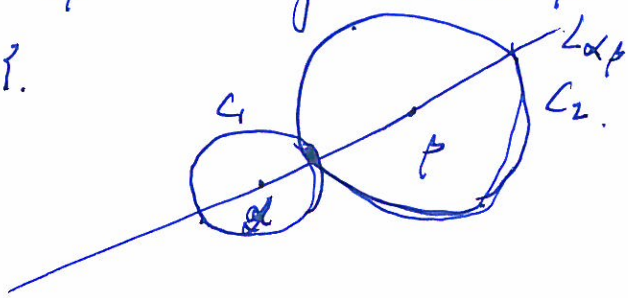
fp must lie on the circle C_2 of radius $R_2 = d(\beta, p)$ centred at β.

So $fp \in C_1 \cap C_2$.

If p is on the line $L_{\alpha\beta}$ connecting α and β

then $C_1 \cap C_2 = \{p\}$.

So $fp = p$ if $p \in L_{\alpha\beta}$



Thus if $f : \mathbb{E}^2 \to \mathbb{E}^2$ is an isometry and

$\alpha, \beta \in \mathbb{E}^2$ are such that

$\alpha \neq \beta$ and $f\alpha = \alpha$ and $f\beta = \beta$

then $f\rho = \rho$ for every $\rho \in L_{\alpha\beta}$

where $L_{\alpha\beta}$ is the line connecting α and β.

If $f : \mathbb{E}^2 \to \mathbb{E}^2$ is an isometry and

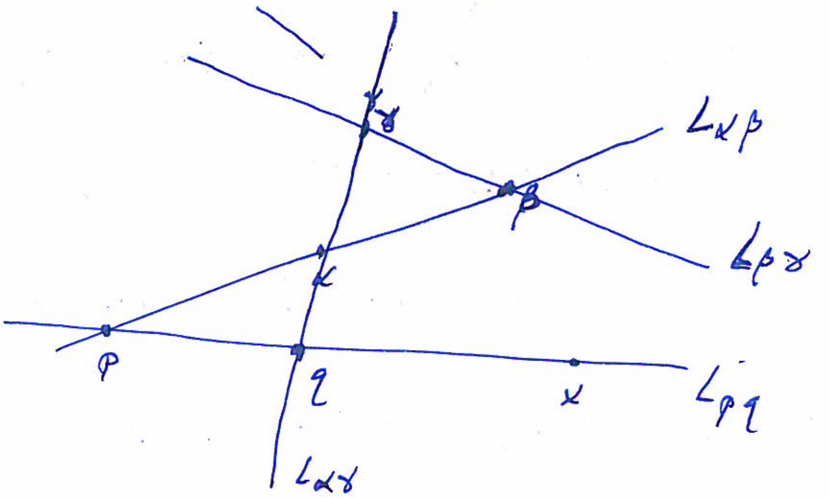
$\gamma, \alpha, \beta \in \mathbb{E}$ are such that $\gamma \notin L_{\alpha\beta}$ and $\alpha \neq \beta$ and

$f\alpha = \alpha$, $f\beta = \beta$ and $f\gamma = \gamma$

then f fixes all of $\mathbb{E}^2$.

Proof f fixes $L_{\alpha\beta}$, $L_{\alpha\gamma}$ and $L_{\beta\gamma}$.

If $p \in L_{\alpha\beta}$ and $q \in L_{\alpha\gamma}$ then f fixes L_{pq}.



Every point $x \in \mathbb{E}^2$ is on some L_{pq} with $p \in L_{\alpha\beta}$ and

$q \in L_{\alpha\gamma}$

and so $fx = x$.

So $f = id_{\mathbb{E}^2}$.