Lecture 30 Proof of the isometry = affine orthogonal group.
Group Theory and linear algebra 14.10.2011

Theorem Define

$$\Phi : AO_n(\mathbb{R}) \longrightarrow Isom(\mathbb{E}^n)$$
$$y \longmapsto f_y \quad , \qquad \text{where}$$

$$f_y\left(\begin{pmatrix} x \\ 1 \end{pmatrix}\right) = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} x \\ 1 \end{pmatrix} \quad \text{if } y = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right).$$

Then Φ is a group isomorphism.

Proof To show: (a) Φ is a function (Φ is well defined).

(b) Φ is a group homomorphism.

(c) Φ is a bijection.

(b) If $y, z \in AO_n(\mathbb{R})$ ~~then~~ and $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{E}^n$ then

$$f_y f_z \begin{pmatrix} x \\ 1 \end{pmatrix} = yz \begin{pmatrix} x \\ 1 \end{pmatrix} = f_{yz}\begin{pmatrix} x \\ 1 \end{pmatrix}.$$

So Φ is a homomorphism.

(a) To show: If $y \in AO_n(\mathbb{R})$ then f_y is an isometry.

Assume
$$y = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right) = X^\mu g \quad \text{with } \mu \in \mathbb{R}^n \text{ and } g \in O_n.$$

Then
$f_y = t_\mu g$ where $t_\mu : \mathbb{E}^n \longrightarrow \mathbb{E}^n$, $\begin{pmatrix} x \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} \mu + x \\ 1 \end{pmatrix}$ is a translation

and $g : \mathbb{E}^n \longrightarrow \mathbb{E}^n$, $\begin{pmatrix} x \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} gx \\ 1 \end{pmatrix}$ with $g \in O_n(\mathbb{R})$ so that $gg^t = 1$.

If $x, z \in \mathbb{E}^n$ then

$$d(t_\mu x, t_\mu z) = d(\mu + x, \mu + z) = \sqrt{\langle (\mu+x)-(\mu+z), (\mu+x)-(\mu+z)\rangle}$$
$$= \sqrt{\langle x-z, x-z\rangle} = d(x,z)$$

and

$$\langle gx, gz\rangle = (x_1, \dots, x_n)\, g^t g \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = (x_1, \dots, x_n) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$
$$= \langle x, z\rangle$$

so that

$$d(gx, gz) = \sqrt{\langle gx - gz, gx - gz\rangle} = \sqrt{\langle g(x-z), g(x-z)\rangle}$$
$$= \sqrt{\langle x-z, x-z\rangle} = d(x,z).$$

Thus g, t_μ and $f_y = t_\mu g$ are all isometries.

(c) To show: There is an inverse function to Φ.

Define

$$\Psi : Isom(\mathbb{E}^n) \longrightarrow AO_n(\mathbb{R})$$
$$f \longmapsto \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right)$$

where

$$\mu = f(0) \quad \text{and}$$

$$g = \begin{pmatrix} | & | & & | \\ g_1 & g_2 & \cdots & g_n \\ | & | & & | \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} | \\ g_j \\ | \end{pmatrix} = f\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} j^{th} = f(e_j)$$

where e_j is $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} j^{th}$

~~(ca)~~ To show: (ca) Ψ is well defined.

(cb) $\Psi \circ \Phi = id_{AO_n}$ and $\Phi \circ \Psi = id_{Isom}$.

(cb) Let $\left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right) \in AO_n$. Then

$$(\Psi \circ \Phi)\left(\begin{array}{c|c} g & \mu \\ \hline & 1 \end{array}\right) = \Psi(f_y) = \left(\begin{array}{c|c} g' & \mu' \\ \hline 0 & 1 \end{array}\right)$$

where $g' = \begin{pmatrix} | & | & & | \\ g_1' & g_2' & \cdots & g_n' \\ | & | & & | \end{pmatrix}$ with $g_j' = f_y(e_j) = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} g_j \\ \hline 1 \end{pmatrix} = \begin{pmatrix} | \\ g_j \\ | \end{pmatrix}$

and $\mu' = f_y(0)$ with $f_y(0) = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mu \\ \hline 1 \end{pmatrix} = \mu$.

So $(\Psi \circ \Phi)\left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right)$.

(cbb) Let $f \in Isom$. Then

$$(\Phi \circ \Psi)(f) = \Phi\left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right) = f_y \quad \text{where } y = \left(\begin{array}{c|c} g & \mu \\ \hline 0 & 1 \end{array}\right)$$

and $g = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix}$ with $f(e_j) = \begin{pmatrix} | \\ g_j \\ | \end{pmatrix} = g\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = ge_j$.

and $\mu = f(0)$.

To show: $f\begin{pmatrix} x \\ \hline 1 \end{pmatrix} = f_y\begin{pmatrix} x \\ \hline 1 \end{pmatrix}$.

To show: $t_{-\mu} f = t_{-\mu} f_y$.

Let $g = t_{-\mu} f_y$.

Let $h = t_{-\mu} f$.

To show: If $x \in \mathbb{E}^n$ then $hx = gx$.

We know $h \in \mathrm{Isom}(\mathbb{E}^n)$ and $h(0) = 0$.

If $x, z \in \mathbb{E}^n$ then, since $h(0) = 0$,

$$\langle hx, hz\rangle = \tfrac{1}{2}\left(\langle hx, hx\rangle + \langle hz, hz\rangle - \langle hx-hz, hx-hz\rangle\right)$$

$$= \tfrac{1}{2}\left(d(hx,0)^2 + d(hz,0)^2 - d(hx,hz)^2\right)$$

$$= \tfrac{1}{2}\left(d(hx,h0)^2 + d(hz,h0)^2 - d(hx,hz)^2\right)$$

$$= \tfrac{1}{2}\left(d(x,0)^2 + d(z,0)^2 - d(x,z)^2\right)$$

$$= \tfrac{1}{2}\left(\langle x,x\rangle + \langle z,z\rangle - \langle x-z, x-z\rangle\right) = \langle x, z\rangle.$$

Assume $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{E}^n$.
Since $he_i = ge_i$,

j^{th} entry of $hx = \langle hx, e_j\rangle$

$$= \langle h(x_1e_1 + \cdots + x_ne_n), e_j\rangle = \langle x_1e_1 + \cdots + x_ne_n, h^{-1}e_j\rangle$$

$$= x_1\langle e_1, h^{-1}e_j\rangle + \cdots + x_n\langle e_n, h^{-1}e_j\rangle$$

$$= x_1\langle he_1, e_j\rangle + \cdots + x_n\langle he_n, e_j\rangle$$

$$= x_1 g_{j1} + \cdots + x_n g_{jn} = j^{\text{th}} \text{ entry of } g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

So $hx = gx$.