Group Theory and Linear algebra, Friday 29 July 2011.
Lecture 3 Equivalence relations.

A set is a collection of elements.

Let S and T be sets. The product of S and T is the set

$$S \times T = \{(s,t) \mid s \in S,\ t \in T\}$$

Example If $S = \{1,2,3\}$ then

$$S \times S = \begin{Bmatrix} (1,1), & (1,2), & (1,3) \\ (2,1), & (2,2), & (2,3) \\ (3,1), & (3,2), & (3,3) \end{Bmatrix}$$

Let S be a set. A relation on S is a subset of $S \times S$.

Examples: (1) $<$ is a relation on $\mathbb{Z}$.

$a < b$ if there exists $x \in \mathbb{Z}_{>0}$ such that $a + x = b$.

$a < b$ means (a,b) is in the relation $<$.

(2) Let $m \in \mathbb{Z}$.

Let $a, b \in \mathbb{Z}$. Define $= \bmod m$, a relation on $\mathbb{Z}$, by

$a = b \bmod m$ if $r_a = r_b$,

where $a = q_a m + r_a$ and $b = q_b m + r_b$

with $q_a, q_b \in \mathbb{Z}$ and $0 \le r_a < |m|$ and $0 \le r_b < |m|$

Define $a \bmod m$ to be r_a,

where $a = q_a m + r_a$ with $q_a \in \mathbb{Z}$ and $0 \le r_a < |m|$.

Let S be a set.
Let $\sim$ be a relation on S.
Write $s_1 \sim s_2$ if (s_1, s_2) is in the relation $\sim$.

The relation $\sim$ is ~~symmetric~~ reflexive if $\sim$ satisfies:

if $s \in S$ then $s \sim s$.

The relation $\sim$ is symmetric if $\sim$ satisfies:

if $s_1, s_2 \in S$ and $s_1 \sim s_2$ then $s_2 \sim s_1$

The relation $\sim$ is transitive if $\sim$ satisfies:

if $s_1, s_2, s_3 \in S$ and $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$

An equivalence relation on S is a relation on S that is reflexive, symmetric and transitive.
Let S be a set.
Let $\sim$ be an equivalence relation on S
Let $s \in S$.

The equivalence class of s is the set

$$[s] = \{ x \in S \mid x \sim s \}.$$

A partition of S is a collection $\mathcal{S}$ of subsets of S such that

(a) $\bigcup_{Y \in \mathcal{S}} Y = S$

(b) If $X, Y \in \mathcal{S}$ and $X \neq Y$ then $X \cap Y = \emptyset$.

Example Let $m=7$.

Then $36 \bmod 7 = 1$, since $36 = 5\cdot 7+1$,

$-6 \bmod 7 = 1$, since $-6 = -1\cdot 7+1$,

$1 \bmod 7 = 1$, since $1 = 0\cdot 7+1$.

The equivalence class of 36 is

$[36] = \{\ldots, -13, -6, 1, 8, 15, 22, 29, 36, \ldots\} = [1]$,

$\{\ldots, -12, -5, 2, 9, 16, 23, 30, 37, \ldots\} = [2]$,

$\{\ldots, -11, -4, 3, 10, 17, 24, \ldots\} = [3]$,

$\{\ldots, -10, -3, 4, 11, 18, 25, \ldots\} = [4]$,

$\{\ldots, -9, -2, 5, 12, 19, 26, \ldots\} = [5]$

$\{\ldots, -8, -1, 6, 13, 20, 27, \ldots\} = [6]$

$\{\ldots, -7, 0, 7, 14, 21, 28, \ldots\} = [7]$.

Recall that $\mathbb{Z}/m\mathbb{Z} = \{1, 2, 3, 4, 5, 6, 7\}$.

Note that

$$\{[1], [2], [3], [4], [5], [6], [7]\}$$

is a partition of $\mathbb{Z}$ since

(a) $[1] \cup [2] \cup [3] \cup [4] \cup [5] \cup [6] \cup [7]$ and

(b) if $i, j \in \{1, \ldots, 7\}$ and $i \neq j$ then

$$[i] \cap [j] = \phi.$$

Theorem Let $m \in \mathbb{Z}$. Then $= \bmod m$ is an equivalence relation on $\mathbb{Z}$.

Proof To show: (a) $= \bmod m$ is reflexive

(b) $= \bmod m$ is symmetric

(c) $= \bmod m$ is transitive.

To show: (a) If $a \in \mathbb{Z}$ then $a = a \bmod m$.

(b) If $a, b \in \mathbb{Z}$ and $a = b \bmod m$ then $b = a \bmod m$

(c) If $a, b, c \in \mathbb{Z}$ and $a = b \bmod m$ and $b = c \bmod m$ then $a = c \bmod m$.

Assume $a, b, c \in \mathbb{Z}$ and $a = b \bmod m$ and $b = c \bmod m$.

Let $a = q_a \cdot m + r_a$, $b = q_b \cdot m + r_b$, $c = q_c \cdot m + r_c$

with $0 \leq r_a < m$, $0 \leq r_b < m$, $0 \leq r_c < m$.

Since $a = b \bmod m$ and $b = c \bmod m$ then

$r_a = r_b$ and $r_b = r_c$.

Since $=$ is an equivalence relation on $\mathbb{Z}$,

$r_a = r_a$, $r_b = r_a$ and $r_a = r_c$.

$\therefore$ $a = a \bmod m$, $b = a \bmod m$ and $a = c \bmod m$.

$\therefore$ $= \bmod m$ is an equivalence relation on $\mathbb{Z}$.

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