Lecture 31 Group Theory and linear algebra 18.10.2011

Theorem

Let G be a finite subgroup of $Isom(\mathbb{E}^2)$.

Then G is a cyclic or a dihedral group.

Proof

Step 1 Let $p \in \mathbb{E}^2$ and $G = \{g_1, \dots, g_r\}$.

Then

$q = g_1 p + \cdots + g_r p$ is a fixed point of G.

So every element of G is a rotation about q or a reflection in a line through q.

Let $h = r_{\theta, q}$ with θ minimum possible and let $H = \langle h \rangle$.

Then H is a cyclic group.

Case 1: If $H = G$ then G is cyclic

Case 2: If $H \neq G$ let

$s_1, s_2 \in G$ such that $s_2, s_1 \notin H$. and $s_1 \neq s_2$ then

$s_1 s_2 \in H$ so $s_1 \in H s_2^{-1} = H s_2$, since $s_2^2 = 1$.

So $G = \{H, s_1 H\}$. with $s_1^2 = 1$.

So G is a dihedral group. //

Groups

A **group** is a set G with a function

$$\begin{array}{rcl} G\times G & \to & G \\ (g_1,g_2) & \mapsto & g_1g_2 \end{array} \quad \text{such that}$$

(a) If $g_1,g_2,g_3\in G$ then $(g_1g_2)g_3 = g_1(g_2g_3)$,

(b) There exists $1\in G$ such that

if $g\in G$ then $g\cdot 1=g$ and $1\cdot g=g$.

(c) If $g\in G$ then there exists $g^{-1}\in G$ such that

$gg^{-1}=1$ and $g^{-1}g=1$.

Homomorphisms are for comparing groups

A **homomorphism from G to H** is a function $f: G\to H$ such that

(a) If $g_1,g_2\in G$ then $f(g_1g_2)=f(g_1)f(g_2)$,

Let $f: G\to H$ be a homomorphism.

The **kernel** of f is

$$\ker f = \{g\in G \mid f(g)=1\}$$

and the **image** of f is

$$\operatorname{im} f = \{f(g) \mid g\in G\}$$

Examples of groups

Cyclic groups, Dihedral groups, Symmetric groups.

$S_n = \{\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid \sigma \text{ is a bijection}\}$

= { graphs with n top vertices and n bottom vertices such that each top dot is connected to exactly one bottom dot and each bottom dot is connected to exactly one top dot. }

with product given by composition $\sigma_1 \sigma_2 = \begin{array}{|c|}\hline \sigma_1 \\ \hline \sigma_2 \\ \hline \end{array}$

S_3 = { |||, X|, |X, ⋇, \X, X/ }

S_2 = { ||, X }

S_1 = { | }

and

	|||	X|	|X	⋇	\X	X/
|||	|||	X|	|X	⋇	\X	X/
X|	X|	|||				
|X	|X		|||			
⋇	⋇			|||		
\X	\X				X/	|||
X/	X/				|||	X/

Note that

$A = \{ ||||, \#, \#, \# \}$ is a subgroup of S_4

$A = \langle \# \rangle$, the group generated by $\#$.

Also

$B = \langle |*, \# \rangle$

$= \left\{ \begin{matrix} ||||, \#, \#, \# \\ |*, *, *|, XX \end{matrix} \right\}$ is a subgroup of S_4.