Lecture 36 $\mathbb{C}[t]$-modules, Group theory and linear algebra

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(1) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 7 \end{pmatrix}$.

A acts on $\mathbb{C}^3 = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{C} \right\}$.

$\mathbb{C}^3$ has basis $\{e_1, e_2, e_3\}$ where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Define an action of $\mathbb{C}[t]$ on $\mathbb{C}^3$ by, for example,

$$(3+4t+5t^2)\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = (3+4A+5A^2)\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

This is the <u>$\mathbb{C}[t]$-module</u> <u>defined by A</u>.

(2) Let $\mathbb{C}[t]^{\oplus 3} = \left\{ \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \mid g_1, g_2, g_3 \in \mathbb{C}[t] \right\}$

so that, for example,

$$\begin{pmatrix} 3t+7 \\ t^3-2t+1 \\ 5t^2+t^6 \end{pmatrix} \in \mathbb{C}[t]^{\oplus 3} \quad \text{and} \quad \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{C}[t]^{\oplus 3}$$

$\mathbb{C}[t]$ acts on $\mathbb{C}[t]^{\oplus 3}$ by, for example,

$$(3+4t+5t^2)\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9+12t+15t^2 \\ 3+4t+5t^2 \\ 6+8t+10t^2 \end{pmatrix}$$

$\mathbb{C}[t]^{\oplus 3}$ has basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}, \begin{pmatrix} t^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ t^2 \end{pmatrix}, \ldots \right\}.$$

$\mathbb{C}[t]^{\oplus 3}$ is infinite dimensional.

$\mathbb{C}[t]^{\oplus 3}$ is <u>the free $\mathbb{C}[t]$-module generated by</u> $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

A <u>$\mathbb{C}[t]$-module</u> is a vector space V over $\mathbb{C}$ with a function

$$\begin{matrix} \mathbb{C}[t] \times V & \to & V \\ (g, v) & \mapsto & gv \end{matrix} \quad \text{such that}$$

(a) If $g_1, g_2 \in \mathbb{C}[t]$ and $v \in V$ then $(g_1 g_2)v = g_1(g_2 v)$,

(b) If $v \in V$ then $1 \cdot v = v$,

(c) If $c_1, c_2 \in \mathbb{C}$, $g_1, g_2 \in \mathbb{C}[t]$ and $v \in V$ then

$$(c_1 g_1 + c_2 g_2)v = c_1 g_1 v + c_2 g_2 v$$

(d) If $c_1, c_2 \in \mathbb{C}$, $g \in \mathbb{C}[t]$ and $v_1, v_2 \in V$ then

$$g(c_1 v_1 + c_2 v_2) = c_1 g v_1 + c_2 g v_2.$$

A <u>$\mathbb{C}[t]$-module homomorphism from V to W</u> is a ~~function~~ linear transformation

$$f : V \to W \quad \text{such that}$$

If ~~$g_1, g_2 \in \mathbb{C}[t]$~~ $g \in \mathbb{C}[t]$ and $v \in V$ then

$$f(gv) = g f(v).$$

The <u>kernel of f</u> is

$$\ker f = \{ v \in V \mid f(v) = 0 \}$$

and the <u>image of f</u> is

$$\operatorname{im} f = \{ f(v) \mid v \in V \}.$$

(3) The map

$$\Phi : \mathbb{C}[t]^{\oplus 3} \longrightarrow \mathbb{C}^3$$

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \longmapsto g_1(A)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + g_2(A)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + g_3(A)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is a $\mathbb{C}[t]$-module homomorphism.

Proposition $\ker \Phi = (t-A)\,\mathbb{C}[t]^{\oplus 3}$

More specifically,

$$t-A = \begin{pmatrix} t-1 & -2 & -3 \\ -4 & t-5 & -6 \\ -9 & -8 & t-7 \end{pmatrix} \quad \text{and}$$

$$\ker \Phi = \left\{ \begin{pmatrix} t-1 & -2 & -3 \\ -4 & t-5 & -6 \\ -9 & -8 & t-7 \end{pmatrix}\begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix} \;\middle|\; g_1(t), g_2(t), g_3(t) \right\}.$$

Consequence (1st homomorphism theorem)

$$\mathbb{C}^3 \cong \frac{\mathbb{C}[t]^{\oplus 3}}{(t-A)\,\mathbb{C}[t]^{\oplus 3}} \quad \text{as } \mathbb{C}[t]\text{-modules.}$$

POINT! A acting on $\mathbb{C}^3$ is very similar to a "clock" $\frac{\mathbb{Z}}{12\mathbb{Z}}$.

Use row reduction to find invertible matrices L and R (with entries in $\mathbb{C}[t]$) such that $L(t-A)R$ is diagonal.

Example $A = \begin{pmatrix} 3 & -7 \\ 1 & 5 \end{pmatrix}$

Then

$$(t-A) = \begin{pmatrix} t-3 & 7 \\ -1 & t-5 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} t-3 & 7 \\ -1 & t-5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & t-5 \\ t-3 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -(t-5) \\ t-3 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & t-5 \\ 0 & 7+(t-5)(t-3) \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & t-5 \\ 0 & t^2-8t+22 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & t^2-8t+22 \end{pmatrix}$$

$t^2-8t+22 = (t-\lambda)(t-\mu)$ where

$$\{\mu, \lambda\} = \left\{ \frac{8 \pm \sqrt{64-44}}{2} \right\} = \left\{ \frac{8 \pm \sqrt{20}}{2} \right\} = \{ 4 \pm \sqrt{5} \}.$$

and

$$L = \begin{pmatrix} 1 & -(t-3) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -(t-3) & -1 \\ 1 & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & -(t-5) \\ 0 & 1 \end{pmatrix}$$

So

$$L(t-A)R = L \begin{pmatrix} t-3 & 7 \\ -1 & t-5 \end{pmatrix} R = \begin{pmatrix} 1 & 0 \\ 0 & t^2-8t+22 \end{pmatrix}$$

and

$$\frac{\mathbb{C}[t]^{\oplus 2}}{(t-A)\mathbb{C}[t]^{\oplus 2}} \longleftrightarrow \frac{L\,\mathbb{C}[t]^{\oplus 2}}{L(t-A)\mathbb{C}[t]^{\oplus 2}} = \frac{\mathbb{C}[t]^{\oplus 2}}{L(t-A)R\,\mathbb{C}[t]^{\oplus 2}}$$

$$\begin{pmatrix} g_1 \\ g_2 \\ \cancel{g_3} \end{pmatrix} \longmapsto L \begin{pmatrix} g_1 \\ g_2 \\ \cancel{g_3} \end{pmatrix}$$

and

$$\frac{\mathbb{C}[t]^{\oplus 2}}{\begin{pmatrix} 1 & 0 \\ 0 & t^2-8t+22 \end{pmatrix}\mathbb{C}[t]^{\oplus 2}} = \frac{\mathbb{C}[t]}{\mathbb{C}[t]} \oplus \frac{\mathbb{C}[t]}{(t^2-8t+22)\mathbb{C}[t]} = \frac{\mathbb{C}[t]}{(t^2-8t+22)\mathbb{C}[t]}$$

$$\frac{\mathbb{C}[t]}{(t^2-8t+22)\,\mathbb{C}[t]} \text{ has basis } \{1, t\}$$

with t-action on this basis given by the matrix

$$\begin{pmatrix} 0 & -22 \\ 1 & 8 \end{pmatrix}$$