Lecture 4 Functions

Functions are for comparing sets.

Let S and T be sets.

A <u>function</u> from S to T is a subset f of $S\times T$,

$$f = \{(s, f(s)) \mid s \in S\},$$

such that

(a) If $s \in S$ then there exists $t \in T$ such that $(s,t) \in f$

(b) If $s \in S$, and $t_1, t_2 \in T$ and $(s,t_1), (s,t_2) \in f$ then $t_1 = t_2$.

The function f is an assignment assigning a mark $f(s)$ from T to each $s \in S$. Write

$$\begin{array}{l} f: S \to T \\ s \mapsto f(s) \end{array} \quad \text{or} \quad S \xrightarrow{f} T.$$

A function $f: S \to T$ is <u>injective</u> if it satisfies:

if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$.

A function $f: S \to T$ is <u>surjective</u> if it satisfies:

if $t \in T$ then there exists $s \in S$ such that $f(s) = t$.

A function $f: S \to T$ is <u>bijective</u> if it is injective and surjective

Let $f: S \to T$ and $g: S \to T$ be functions.
The functions f and g are *equal* if they satisfy
if $s \in S$ then $f(s) = g(s)$.

Let $f: S \to T$ and $g: T \to U$ be functions
The *composition of g and f* is the function
$g \circ f : S \to T$ given by $(g \circ f)(s) = g(f(s))$.

Let S be a set.
The *identity function on S* is the function
$id_S : S \to S$ given by $id_S(s) = s$.

Let $f: S \to T$ be a function.
An *inverse function* to $f: S \to T$ is a function
$g: T \to S$ such that

$$g \circ f = id_S \quad \text{and} \quad f \circ g = id_T.$$

Theorem Let $f: S \to T$ be a function.

(a) An inverse function to f exists if and only if f is bijective.

(b) If an inverse function to f exists then it is unique.

Proof Assume $f: S \to T$ is a function.

(a) To show: An inverse function to f exists if and only if f is bijective.

$\Rightarrow$ Assume that an inverse function to f exists: $g: T \to S$ such that $g \circ f = id_S$ and $f \circ g = id_T$.

To show: f is bijective.

To show: (1) f is injective
(2) f is surjective.

(1) To show: If $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$ then $s_1 = s_2$.

Assume $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$.

To show: $s_1 = s_2$

$$s_1 = id_S(s_1) = (g \circ f)(s_1) = g(f(s_1))$$
$$= g(f(s_2)) = (g \circ f)(s_2) = id_S(s_2) = s_2.$$

(2) To show: If $t \in T$ then there exists $s \in S$ such that $f(s) = t$.

Assume $t \in T$

To show: There exists $s \in S$ such that $f(s) = t$.

Let $s = g(t)$

To show: $f(s) = t$.

$$f(s) = f(g(t)) = (f \circ g)(t) = id_T(t) = t.$$

So f is bijective.

$\Leftarrow$ To show: If $f: S \to T$ is bijective then an inverse function $g: T \to S$ exists.

Assume $f: S \to T$ is bijective.

To show: There exists $g: T \to S$ such that
$$g \circ f = id_S \text{ and } f \circ g = id_T$$

Let $g: T \to S$ be given by
$$g = \{(t, s) \in T \times S \mid f(s) = t\}.$$

To show: (a) g is a function

(b) $g \circ f = id_S$

(c) $f \circ g = id_T$

(a) To show: (aa) If $t \in T$ then there exists $s \in S$ such that $(t, s) \in g$.

(ab) If $t \in T$ and $s_1, s_2 \in S$ and $(t, s_1) \in g$ and $(t, s_2) \in g$ then $s_1 = s_2$.

(aa) Assume $t \in T$.

To show: There exists $s \in S$ such that $(t, s) \in g$.

To show: There exists $s \in S$ such that $f(s) = t$.

This holds since f is surjective.

(ab) Assume $t \in T$, $s_1, s_2 \in S$ and $(t, s_1), (t, s_2) \in g$.

To show: $s_1 = s_2$.

Since $(t, s_1), (t, s_2) \in g$, then $f(s_1) = t$ and $f(s_2) = t$.

Since f is injective, $s_1 = s_2$.

(b) To show: $g \circ f = id_S$

To show: If $s \in S$ then $(g \circ f)(s) = id_S(s)$.

Assume $s \in S$.

To show: $(g \circ f)(s) = id_S(s)$.

$(g \circ f)(s) = g(f(s)) = s_1$, where $s_1 \in S$ such that $f(s_1) = \cancel{f(t)}f(s)$

Since f is injective, $s_1 = s$.

So $(g \circ f)(s) = s = id_S(s)$.

(c) To show: $(f \circ g) = id_T$

To show: If $t \in T$ then $(f \circ g)(t) = id_T(t)$.

Assume $t \in T$

To show: $(f \circ g)(t) = id_T(t)$

$(f \circ g)(t) = f(g(t)) = f(s)$, where $s \in S$ such that $f(s) = t$.

So $(f \circ g)(t) = f(s) = t = id_T(t)$.

So g is an inverse function to f.

<u>Part (b) of the Theorem</u>

To show: The inverse function to $f: S \to T$ is unique.

Assume $g_1: T \to S$ and $g_2: T \to S$ are inverse functions to f.

To show: $g_1 = g_2$

To show: If $t \in T$ then $g_1(t) = g_2(t)$.

We know that

$$(f \circ g_1) = id_T, \quad (g_1 \circ f) = id_S, \quad (f \circ g_2) = id_T, \quad (g_2 \circ f) = id_S.$$

Assume $t \in T$.

To show: $g_1(t) = g_2(t)$.

$$g_1(t) = g_1(id_T(t)) = g_1((f \circ g_2)(t)) = g_1(f(g_2(t)))$$
$$= (g_1 \circ f)(g_2(t)) = id_S(g_2(t)) = g_2(t).$$

So $g_1 = g_2$

So the inverse function to $f: S \to T$ is unique.//