Group Theory and Linear Algebra 09.08.2011

Let $\mathbb{F}$ be a field. Lecture 7: Vector spaces

A vector space over $\mathbb{F}$ is an abelian group V with a function

$$\mathbb{F} \times V \to V$$
$$(c, v) \mapsto cv \quad \text{such that}$$

(a) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $c_1(c_2 v) = (c_1 c_2) v$,

(b) If $v \in V$ then $1v = v$

(c) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $(c_1 + c_2)v = c_1 v + c_2 v$

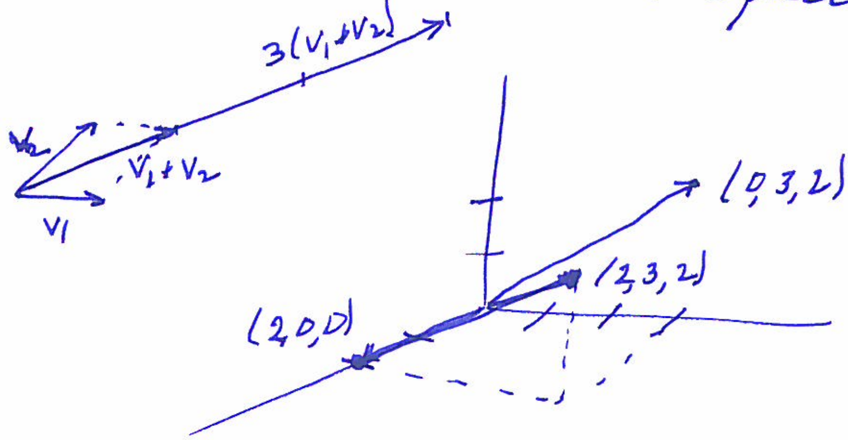
(d) If $c \in \mathbb{F}$ and $v_1, v_2 \in V$ then $c(v_1 + v_2) = cv_1 + cv_2$.

Examples: (1) Column vectors of length 3, is a vector space over $\mathbb{F}$,

$$\mathbb{F}^3 = M_{3\times 1}(\mathbb{F}) = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \middle| \, a_i \in \mathbb{F} \right\} \text{ with}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \text{ and } c\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix}$$

(2) Vectors in $\mathbb{R}^3$ is a vector space over $\mathbb{R}$,



Let V be a vector space over $\mathbb{F}$.

A <u>subspace of V</u> is a subset $U \subseteq V$ such that

(a) If $u_1, u_2 \in U$ then $u_1 + u_2 \in U$.

(b) If $u \in U$ then $-u \in U$.

(c) $0 \in U$

(d) If $u \in U$ and $c \in \mathbb{F}$ then $cu \in U$.

<u>Proposition</u> Let V be a vector space over $\mathbb{F}$. Let U and W be subspaces of V.

Then

$$U \cap W = \{ v \in V \mid v \in U \text{ and } v \in W \} \quad \text{and}$$

$$U + W = \{ u + w \mid u \in U \text{ and } w \in W \}$$

are subspaces of V.

Let V be a vector space over $\mathbb{F}$.

Let U, W be subspaces of V.

The subspaces U, W are <u>complementary</u> if they satisfy

(a) $U \cap W = \{0\}$

(b) $U + W = V$.

Write $V = U \oplus W$ if U and W are complementary subspaces of V.

Linear transformations are for comparing vector spaces.

Let $\mathbb{F}$ be a field and let A and V be vector spaces over $\mathbb{F}$.

A <u>linear</u> <u>transformation</u> from A to V is a function $f: A \to V$ such that

(a) If $a_1, a_2 \in A$ then $f(a_1+a_2) = f(a_1)+f(a_2)$,

(b) If $c \in \mathbb{F}$ and $a \in A$ then $f(ca) = cf(a)$.

<u>Proposition</u> Let $f: A \to V$ be a linear transformation. Then

(a) $f(0) = 0$ and (b) if $a \in A$ then $f(-a) = -f(a)$.

<u>Proof</u> Assume $f: A \to V$ is a linear transformation.

To show: (a) $f(0) = 0$.

(b) If $a \in A$ then $f(-a) = -f(a)$.

(a) To show: $f(0) = 0$.

$f(0) = f(0+0) = f(0) + f(0)$.

Add $-f(0)$ to each side.

So $0 = f(0)$.

(b) Assume $a \in A$.

To show: $f(-a) = -f(a)$.

To show: (ba) $f(a) + f(-a) = 0$

(bb) $f(-a) + f(a) = 0$

(ba) $f(a)+f(-a) = f(a+(-a)) = f(0) = 0$.

(bb). This follows from (a) since V is a commutative group.

Let $f: A \to V$ be a linear transformation.

The <u>kernel of f</u>, ~~or~~ <u>null space</u>, of f

$$\ker f = \{a \in A \mid f(a) = 0\}$$

The <u>image of f</u> is

$$\operatorname{im} f = \{f(a) \mid a \in A\}.$$

<u>Proposition</u> Let $f: A \to V$ be a linear transformation

(a) $\ker f$ is a subspace of A

(b) $\operatorname{im} f$ is a subspace of V

(c) $\ker f = \{0\}$ if and only if f is injective.

(d) $\operatorname{im} f = V$ if and only if f is surjective.

<u>Proof</u> (a) To show: $\ker f$ is a subspace of A.

To show: (aa) If $a_1, a_2 \in \ker f$ then $a_1 + a_2 \in \ker f$.

(ab) $0 \in \ker f$

(ac) If $a \in \ker f$ then $-a \in \ker f$

(ad) If $c \in \mathbb{F}$ and $a \in \ker f$ then $ca \in \ker f$.

(aa) Assume $a_1, a_2 \in \ker f$.

To show: $a_1 + a_2 \in \ker f$.

To show: $f(a_1 + a_2) = 0$

We know: $f(a_1) = 0$ and $f(a_2) = 0$.

So $f(a_1 + a_2) = f(a_1) + f(a_2) = 0 + 0 = 0$.

(ab) To show: $0 \in \ker f$

To show: $f(0) = 0$.

$f(0) = f(0+0) = f(0) + f(0)$.

Add $-f(0)$ to each side.

So $0 = f(0)$.

(ac) To show: If $a \in \ker f$ then $-a \in \ker f$.

Assume $a \in \ker f$.

To show: $-a \in \ker f$.

To show: $f(-a) = 0$.

We know: $f(a) = 0$.

$f(-a) = f((-1) \cdot a) = (-1) f(a) = (-1) \cdot 0 = 0$.

(ad) To show: if $c \in \mathbb{F}$ and $a \in \ker f$ then $ca \in \ker f$.

Assume $c \in \mathbb{F}$ and $a \in \ker f$

To show: $ca \in \ker f$

To show: $f(ca) = 0$.

We know: $f(a) = 0$

$f(ca) = c f(a) = c \cdot 0 = 0$.

So $\ker f$ is a subspace of A.

(c) To show: $\ker f = \{0\}$ if and only if f is injective.

To show: (ca) If $\ker f = \{0\}$ then f is injective.

(cb) If f is injective then $\ker f = \{0\}$.

(ca) Assume $\ker f = \{0\}$.

To show: f is injective.

To show: If $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$

then $a_1 = a_2$

Assume $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$.

To show: $a_1 = a_2$.

To show: $a_1 - a_2 = 0$.

Since

$$\begin{aligned} f(a_1 - a_2) &= f(a_1 + (-1)a_2) \\ &= f(a_1) + f((-1)a_2) \\ &= f(a_1) + (-1)f(a_2) \\ &= f(a_1) - f(a_2) \\ &= f(a_1) - f(a_1) \qquad (\text{since } f(a_1) = f(a_2)) \\ &= 0, \end{aligned}$$

then $a_1 - a_2 \in \ker f$.

Since $\ker f = \{0\}$, then $a_1 - a_2 = 0$.

So f is injective.

(cb) To show: If f is injective then $\ker f = \{0\}$.

Assume f is injective

To show: $\ker f = \{0\}$

To show: If $a \in A$ and $f(a) = 0$ then $a = 0$.

Assume $a \in A$ and $f(a) = 0$.

To show: $a = 0$

We know ~~then~~ $f(a) = f(0)$.

Since f is injective, $a = 0$.