Group Theory and linear algebra 10.08.2011

Lecture 8: Bases and dimension.

Let $\mathbb{F}$ be a field.

Let V be a vector space over $\mathbb{F}$

Let S be a subset of V.

A linear combination of elements of S is

$$c_1 s_1 + c_2 s_2 + \cdots + c_k s_k, \quad \text{with } k \in \mathbb{Z}_{>0}$$
$$c_1, c_2, \ldots, c_k \in \mathbb{F}$$
$$s_1, s_2, \ldots, s_k \in S.$$

The span of S is

$$\mathrm{span}(S) = \left\{ c_1 s_1 + \cdots + c_k s_k \;\middle|\; \begin{matrix} k \in \mathbb{Z}_{>0},\ c_1, \ldots, c_k \in \mathbb{F} \\ s_1, s_2, \ldots, s_k \in S \end{matrix} \right\}$$

The set S is linearly independent if S satisfies:

If $k \in \mathbb{Z}_{>0}$, $c_1, \ldots, c_k \in \mathbb{F}$, $s_1, s_2, \ldots, s_k \in S$ and
$c_1 s_1 + c_2 s_2 + \cdots + c_k s_k = 0$

then $c_1 = 0$ and $c_2 = 0$ and $c_3 = 0$ and $\ldots$ and $c_k = 0$.

A basis of V is a subset $B \subseteq V$ such that

(a) $\mathrm{span}(B) = V$

(b) B is linearly independent.

The dimension of V is $\mathrm{Card}(B)$, the number of elements in B, for a basis B of V.

<u>Example</u> $\mathbb{R}[t] = \left\{ c_0 + c_1 t + c_2 t^2 + \cdots \;\middle|\; \begin{array}{l} c_i \in \mathbb{R} \text{ and all but a} \\ \text{finite number of } c_i \text{ equal } 0 \end{array} \right\}$

is a vector space over $\mathbb{R}$:

$$(3+4t+t^2) + (0+7t+1t^2) = 3+11t+2t^2$$

and $$8.6(1+2t^2) = 8.6 + 17.2t^2.$$

Let $S = \{1, t, t^2, t^3, \ldots\}$. Then

$\mathrm{Span}(S) = \mathbb{R}[t]$ and S is linearly independent

since

(a) If $p \in \mathbb{R}[t]$ then

$$p = p_0 + p_1 t + p_2 t^2 + \cdots + p_N t^N$$

with $N \in \mathbb{Z}_{>0}$ and $1, t, t^2, \ldots, t^N \in S$ and $p_0, p_1, \ldots, p_N \in \mathbb{R}$,

(b) If $p = p_0 + p_1 t + \cdots + p_N t^N = 0$ then

$p_0 = 0$ and $p_1 = 0$ and $\cdots$ and $p_N = 0$.

Since $\mathrm{Card}(S) = \infty$, the dimension of $\mathbb{R}[t]$ is ∞. //

Let V be a vector space over $\mathbb{F}$.

Let $f: V \to V$ be a linear transformation.

Let $B = \{b_1, b_2, \ldots\}$ be a basis of V.

The <u>matrix of f with respect to B</u> is

$B_f = (f_{ij})$ given by $f(b_j) = f_{1j}b_1 + f_{2j}b_2 + \cdots$

<u>Example</u> Let $\mathbb{R}[t]_{\leq 3} = \{c_0 + c_1 t + c_2 t^2 \mid c_0, c_1, c_2 \in \mathbb{R}\}$

Then $\mathcal{B} = \{1, t, t^2\}$ is a basis of $\mathbb{R}[t]_{\leq 3}$.

Let $f : \mathbb{R}[t]_{\leq 3} \longrightarrow \mathbb{R}[t]_{\leq 3}$ be given by

$$f(p) = p(t-3).$$

For example, $f(3+2t-5t^2) = 3 + 2(t-3) - 5(t-3)^2$

$$= 3 + 2t - 6 - 5(t^2 - 6t + 9)$$
$$= 3 + 2t - 6 - 5t^2 + 30t - 45$$
$$= -48 + 32t - 5t^2.$$

To calculate the matrix of f with respect to $\mathcal{B}$,

$$f(1) = 1 = 1 + 0t + 0t^2$$
$$f(t) = t - 3 = -3 + t + 0t^2$$
$$f(t^2) = (t-3)^2 = t^2 - 6t + 9 = 9 - 6t + t^2$$

and

$$\mathcal{B}_f = \begin{pmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that

$f(3+2t-5t^2) = -48 + 32t - 5t^2$ and

$$\begin{pmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 3-6-45 \\ 2+30 \\ -5 \end{pmatrix} = \begin{pmatrix} -48 \\ 32 \\ -5 \end{pmatrix}.$$

Let $f: V \to V$ be a linear transformation

The kernel, or nullspace, of f is

$$\ker f = \{ v \in V \mid f(v) = 0 \}$$

and the nullity of f is $\dim(\ker f)$.

The image of f is

$$\operatorname{im} f = \{ f(v) \mid v \in V \}$$

and the rank of f is $\dim(\operatorname{im} f)$.

Example Let $f: \mathbb{R}[t] \to \mathbb{R}[t]$ be given by

$$f(p) = tp.$$

For example, $f(1+3t+2t^2) = t+3t^2+2t^3$.

Then

$$\begin{aligned} f(1) &= t \\ f(t) &= t^2 \\ f(t^2) &= t^3 \\ &\vdots \end{aligned}$$

and the matrix of f with respect to the basis $\mathcal{B} = \{1, t, t^2, \dots\}$

is

$$\mathcal{B}_f = \begin{pmatrix} 0 & 0 & 0 & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Also

$\ker f = \{0\}$

$\operatorname{im} f = \operatorname{span}\{t, t^2, t^3, \dots\}$.

So f is injective but not surjective

Let $d : \mathbb{R}[t] \to \mathbb{R}[t]$ be given by

$$d(p) = \frac{d}{dt} \cdot p$$

For example $d(1+3t+2t^2) = 3+4t$.

Then
$d(1) = 0,$
$d(t) = 1,$
$d(t^2) = 2t,$
$d(t^3) = 3t^2,$
$\vdots$

and the matrix of d with respect to the basis $B = \{1, t, t^2, t^3, \ldots\}$

is

$$B_d = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Also

$\ker d = \{c_0 \mid c_0 \in \mathbb{R}\} = \operatorname{span}\{1\}.$

$\operatorname{im} d = \mathbb{R}[t],$

since $d(p_0 t + \frac{p_1}{2} t^2 + \frac{p_2}{3} t^3 + \ldots) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots$

So d is not injective,
d is surjective,
the nullity of d is 1, the rank of d is ∞.