Group Theory and linear algebra 12.08.2011

Lecture 9: Change of basis. (1)

Let V be a vector space.

Let $B = \{b_1, b_2, \ldots\}$ be a basis of V.

Let $C = \{c_1, c_2, \ldots\}$ be another basis of V.

The change of basis matrix from B to C is

$P = (p_{ij})$ given by $c_j = p_{1j} b_1 + p_{2j} b_2 + \cdots$

The change of basis matrix from C to B is

$Q = (q_{ij})$ given by $b_j = q_{1j} c_1 + q_{2j} c_2 + \cdots$

Let $f: V \to V$ be a linear transformation.

The matrix of f with respect to B is

$B_f = (f_{ij}^B)$ given by $f(b_j) = f_{1j}^B b_1 + f_{2j}^B b_2 + \cdots$

The matrix of f with respect to C is

$C_f = (f_{ij}^C)$ given by $f(c_j) = f_{1j}^C c_1 + f_{2j}^C c_2 + \cdots$

Theorem (a) $P = Q^{-1}$.

(b) $B_f = Q C_f Q^{-1}$.

Example

My favourite vector space

$$V = \mathbb{C}^3 = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \;\middle|\; a_1, a_2, a_3 \in \mathbb{C} \right\}$$

has basis

$$\mathcal{B} = \{b_1, b_2, b_3\} \quad \text{with} \quad b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; b_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The matrix

$$B_f = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{defines a linear transformation}$$

$f: V \to V$,

$$f(b_1) = b_2, \quad f(b_2) = b_3, \quad f(b_3) = b_1$$

and $f\begin{pmatrix} 3 \\ 6 \\ 21 \end{pmatrix} = f(3b_1 + 6b_2 + 21b_3) = 3f(b_1) + 6f(b_2) + 21f(b_3)$

$$= 3b_2 + 6b_3 + 21b_1 = \begin{pmatrix} 21 \\ 3 \\ 6 \end{pmatrix}.$$

Another basis of V is

$$C = \{c_1, c_2, c_3\} \quad \text{with} \quad c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad c_2 = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{3}i}{2} \\ \frac{-1-\sqrt{3}i}{2} \end{pmatrix} \quad c_3 = \begin{pmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{2} \end{pmatrix}$$

Then

$$c_1 = b_1 + b_2 + b_3$$

$$c_2 = b_1 + \left(\frac{-1+\sqrt{3}i}{2}\right) b_2 + \left(\frac{-1-\sqrt{3}i}{2}\right) b_3$$

$$c_3 = b_1 + \left(\frac{-1-\sqrt{3}i}{2}\right) b_2 + \left(\frac{-1+\sqrt{3}i}{2}\right) b_3$$

and

$$b_1 = \frac{1}{3}(c_1 + c_2 + c_3)$$

$$b_2 = \frac{1}{3}\left(c_1 + \frac{-1-\sqrt{3}i}{2} c_2 + \frac{-1+\sqrt{3}i}{2} c_3\right)$$

$$b_3 = \frac{1}{3}\left(c_1 + \frac{-1+\sqrt{3}i}{2} c_2 + \frac{-1-\sqrt{3}i}{2} c_3\right)$$

<u>Helpful:</u> Let $\zeta = \frac{-1+\sqrt{3}i}{2}$ and note that

$$\zeta^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 = \frac{1-2\sqrt{3}i-3}{4} = \frac{-1-\sqrt{3}i}{2} \quad \text{and}$$

$$\zeta^3 = \frac{(-1-\sqrt{3}i)}{2}\,\frac{(-1+\sqrt{3}i)}{2} = \frac{1+3}{4} = 1, \text{ and } 1+\zeta+\zeta^2 = 0$$

So

$$c_1 = b_1 + b_2 + b_3$$

$$c_2 = b_1 + \zeta b_2 + \zeta^2 b_3$$

$$c_3 = b_1 + \zeta^2 b_2 + \zeta b_3$$

$$\tfrac{1}{3}(c_1 + c_2 + c_3) = \tfrac{1}{3}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\ \zeta\\ \zeta^2\end{pmatrix} + \begin{pmatrix}1\\ \zeta^2\\ \zeta\end{pmatrix}\right) = \tfrac{1}{3}\begin{pmatrix}3\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} = b_1$$

$$\tfrac{1}{3}(c_1 + \zeta^2 c_2 + \zeta c_3) = \tfrac{1}{3}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \zeta^2\begin{pmatrix}1\\ \zeta\\ \zeta^2\end{pmatrix} + \zeta\begin{pmatrix}1\\ \zeta^2\\ \zeta\end{pmatrix}\right) = \tfrac{1}{3}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}\zeta^2\\ 1\\ \zeta\end{pmatrix} + \begin{pmatrix}\zeta\\ 1\\ \zeta^2\end{pmatrix}\right) = \tfrac{1}{3}\begin{pmatrix}0\\3\\0\end{pmatrix} = b_2$$

$$\tfrac{1}{3}(c_1 + \zeta c_2 + \zeta^2 c_3) = \tfrac{1}{3}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \zeta\begin{pmatrix}1\\ \zeta\\ \zeta^2\end{pmatrix} + \zeta^2\begin{pmatrix}1\\ \zeta^2\\ \zeta\end{pmatrix}\right) = \tfrac{1}{3}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}\zeta\\ \zeta^2\\ 1\end{pmatrix} + \begin{pmatrix}\zeta^2\\ \zeta\\ 1\end{pmatrix}\right) = \tfrac{1}{3}\begin{pmatrix}0\\0\\3\end{pmatrix} = b_3.$$

The change of basis matrix from B to C is

$$P = \begin{pmatrix}1 & 1 & 1\\ 1 & \zeta & \zeta^2\\ 1 & \zeta^2 & \zeta\end{pmatrix}$$

and the change of basis matrix from C to B is

$$Q = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3}\zeta^2 & \frac{1}{3}\zeta\\ \frac{1}{3} & \frac{1}{3}\zeta & \frac{1}{3}\zeta^2\end{pmatrix}$$

and

$$PQ = \begin{pmatrix}1 & 1 & 1\\ 1 & \zeta & \zeta^2\\ 1 & \zeta^2 & \zeta\end{pmatrix}\begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3}\zeta^2 & \frac{1}{3}\zeta\\ \frac{1}{3} & \frac{1}{3}\zeta & \frac{1}{3}\zeta^2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}. \text{ So } P^{-1} = Q.$$

The matrix of f with respect to C:

$$f(c_1) = f\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} = c_1$$

$$f(c_2) = f\begin{pmatrix}1\\\zeta\\\zeta^2\end{pmatrix} = \begin{pmatrix}\zeta^2\\1\\\zeta\end{pmatrix} = \zeta^2\begin{pmatrix}1\\\zeta\\\zeta^2\end{pmatrix} = \zeta^2 c_2.$$

$$f(c_3) = f\begin{pmatrix}1\\\zeta^2\\\zeta\end{pmatrix} = \begin{pmatrix}\zeta\\1\\\zeta^2\end{pmatrix} = \zeta\begin{pmatrix}1\\\zeta^2\\\zeta\end{pmatrix} = \zeta c_3$$

So the matrix of f with respect to C is

$$C_f = \begin{pmatrix}1 & 0 & 0\\0 & \zeta^2 & 0\\0 & 0 & \zeta\end{pmatrix}$$

<u>Magic:</u>

$$Q B_f P = \begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3\zeta^2} & \frac{1}{3\zeta}\\ \frac{1}{3} & \frac{1}{3\zeta} & \frac{1}{3\zeta^2}\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\1 & \zeta & \zeta^2\\1 & \zeta^2 & \zeta\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\1 & \zeta^2 & \zeta\\1 & \zeta & \zeta^2\end{pmatrix}\begin{pmatrix}1 & \zeta^2 & \zeta\\1 & 1 & 1\\1 & \zeta & \zeta^2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 0 & 0\\0 & 3\zeta^2 & 0\\0 & 0 & 3\zeta\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0\\0 & \zeta^2 & 0\\0 & 0 & \zeta\end{pmatrix} = C_f.$$

<u>Theorem</u> Let $f: V \to V$ be a linear transformation. Let B and C be bases of V. Let
P be the change of basis matrix from B to C
Q the change of basis matrix from C to B

B_f the matrix of f with respect to B,

C_f the matrix of f with respect to C.

Then (a) $P = Q^{-1}$

(b) $B_f = Q C_f Q^{-1}$.

Proof (a) To show: (aa) $PQ = 1$

(ab) $QP = 1$.

(aa) We know: $P = (p_{ij})$ with $c_j = p_{1j} b_1 + p_{2j} b_2 + \cdots$

$Q = (q_{k\ell})$ with $b_\ell = q_{1\ell} c_1 + q_{2\ell} c_2 + \cdots$

To show: $PQ = 1$.

To show: (aaa) $(PQ)_{ii} = 1$

(aab) $(PQ)_{ij} = 0$ if $i \neq j$

(aaa) $(PQ)_{ii} = p_{i1} q_{1i} + p_{i2} q_{2i} + p_{i3} q_{3i} + \cdots$

Since

$$\begin{aligned} b_i &= q_{1i} c_1 + q_{2i} c_2 + q_{3i} c_3 + \cdots \\ &= q_{1i}(p_{11} b_1 + p_{21} b_2 + p_{31} b_3 + \cdots) \\ &\quad + q_{2i}(p_{12} b_1 + p_{22} b_2 + p_{32} b_3 + \cdots) \\ &\quad + q_{3i}(p_{13} b_1 + p_{23} b_2 + p_{33} b_3 + \cdots) \\ &\quad \vdots \\ &= (q_{1i} p_{11} + q_{2i} p_{12} + q_{3i} p_{13} + \cdots) b_1 \\ &\quad + (q_{1i} p_{21} + q_{2i} p_{22} + q_{3i} p_{32} + \cdots) b_2 + \cdots \end{aligned}$$

If we use sum notation

$$b_j = \sum_{\ell} q_{\ell j} c_\ell$$

$$= \sum_{\ell} q_{\ell j} \left(\sum_m p_{m\ell} b_m \right)$$

$$= \sum_{\ell, m} p_{m\ell} q_{\ell j} b_m .$$

So $\sum_{\ell} p_{m\ell} q_{\ell j} = 0$ if $m \neq j$ and

$\sum_{\ell} p_{j\ell} q_{\ell j} = 1.$