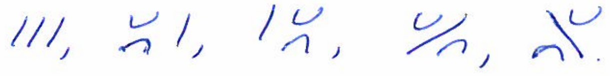
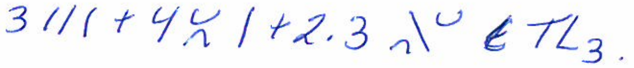
Lecture 1

A <u>vector space</u> is a set of linear combinations of a basis,

$$V = \text{span}\{b_1, \dots, b_n\}.$$

<u>Example:</u> TL_3 has basis

 and



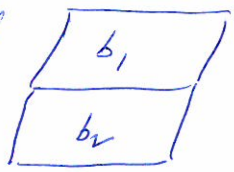
An <u>algebra</u> is a vector space A with a product $A \otimes A \longrightarrow A$ such that

(a) $(a_1 a_2) a_3 = a_1 (a_2 a_3)$ for $a_1, a_2, a_3 \in A$, and

(b) There exists $1 \in A$ such that

$$1a = a1 = a, \quad \text{for } a \in A.$$

<u>Example</u> Define

$$b_1 b_2 = (q+q^{-1})^{\#\text{ of internal loops}}$$ 

An <u>A-module</u> is a vector space M with an <u>action of A</u> $A \otimes M \longrightarrow M$ such that

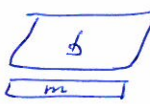
(a) $a_1(a_2 m) = (a_1 a_2) m$, for $a_1, a_2 \in A$, $m \in M$,

(b) $1 \cdot m$, for all $m \in M$

Note: $\otimes$ means that we require the distributive laws.

Example Let M be the vector space with basis

 with

$bm =$  $(q+q^{-1})^{\#\text{ of internal loops}}$.

A representation of A is an A-module M.

Given M define

$$\rho : A \longrightarrow End(M)$$
$$a \longmapsto a_M$$

where a_M is the matrix describing the action of A on M.

The map ρ is a homomorphism of algebras.

Categories

A category $\mathcal{C}$

objects $M \in \mathcal{C}$, morphisms $Hom(M, N)$.

Examples

- Vector spaces Morphisms: Linear transformations
- Algebras Morphisms: Algebra homomorphisms
- A-modules Morphisms: A-module homomorphisms
- Sets Morphisms: functions
-

A <u>simple</u> module is a module M such that if N is a submodule of M then

$$N = M \quad \text{or} \quad N = 0.$$

A module is <u>decomposable</u> if

$$M \simeq N \oplus P$$

Definition: $N \oplus P$ has basis

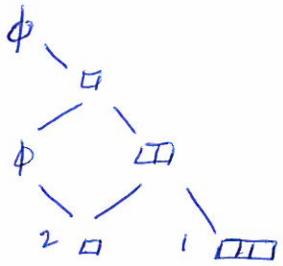
$\{n_1, \dots, n_r, p_1, \dots, p_s\}$ with action

$b_i n_j$ and $b_i p_k$ determined by N and P. Namely,

$$\begin{aligned} A \otimes (N \oplus P) &\longrightarrow N \oplus P \\ an &\longmapsto an \\ ap &\longmapsto ap. \end{aligned}$$

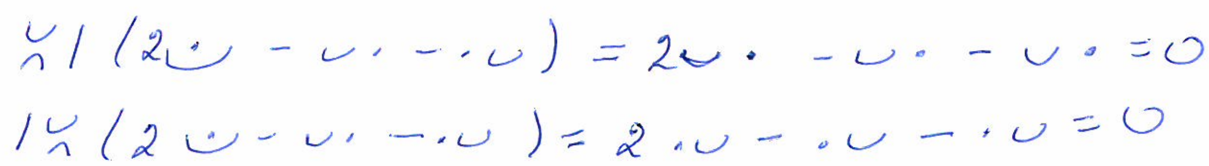
<u>Problem 1</u> Classify the simple modules

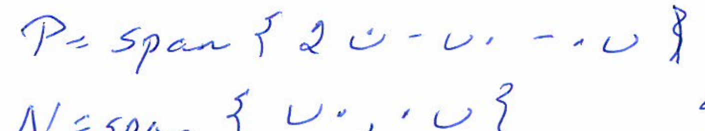
<u>Problem 2</u> Classify the indecomposable modules.



$2^2 + 1^2 = 5$

Consider 

Then 

So 

are submodules and $M = N \oplus P$

P is a simple module since $\dim(P)=1$.

Assume $Q \subseteq \cancel{N}$ is a submodule

Assume $Q \neq 0$. Let $a \cup\cdot + b\cdot\cup \in Q$.

Then $\overset{\cup}{\cap}|\cdot(a\cup\cdot + b\cdot\cup) = a(q+q^{-1})\cup\cdot + b\cup\cdot$

$|\overset{\cup}{\cap}(a\cup\cdot + b\cdot\cup) = a\cdot\cup + b(q+q^{-1})\cdot\cup$

So $\cup\cdot \in Q$ if $a(q+q^{-1})+b \neq 0$

and $\cdot\cup \in Q$ if $a+b(q+q^{-1}) \neq 0$.

If $a(q+q^{-1})+b \neq 0$ and $a+b(q+q^{-1})=0$ then

$N = \text{span}\{\cup\cdot, \cdot\cup\}$ and $P = \text{span}\{p\}$

with $|||p = p$, $\overset{\cup}{\cap}|p = 0$, $|\overset{\cup}{\cap}p = 0$

$p = 2\dot{\cup} - \cup\cdot - \cdot\cup$

are simple modules.

If $a(q+q^{-1})+b=0$ and $a+b(q+q^{-1})=0$ then

$b=-a(q+q^{-1})$ and $1-(q+q^{-1})^2=0$.

So that $(q+q^{-1})^2=1$ or $q^2+1+q^{-2}=0$.

So $(q+q^{-1})=\pm 1$ or $q=e^{\pm 2\pi i/3}$.

<u>Generators and relations</u> Lecture 1 28.07.08. ⑤

~~Let~~ $\cancel{e_1 = \overset{\cup}{\cap}}$. Let A be the algebra given by generators e_1, e_2 and relations

$$e_1e_2e_1 = e_1 \quad \text{and} \quad e_2e_1e_2 = e_2, \quad e_1^2 = (q+q^{-1})e_1$$
$$e_2^2 = (q+q^{-1})e_2$$

Then A contains

$$1,\ e_1,\ e_2,\ e_1e_2,\ e_2e_1,\ \cancel{e_1e_2e_1},\ \cancel{e_2e_1e_2}.$$

and the mutiplication is determined.

Then

$$\begin{aligned} A &\longrightarrow TL_3 \\ e_1 &\longmapsto \overset{\cup}{\cap}\,| \\ e_2 &\longmapsto |\,\overset{\cup}{\cap} \end{aligned}$$

is an algebra isomorphism.

<u>The Regular representation</u>

A is a vector space and A acts on A by multiplication.

A submodule of A is a <u>left ideal</u> of A

<u>Example</u>

~~Ex~~ $TL_3 = \mathrm{span}\{\,|||,\ \overset{\cup}{\cap}\,|,\ |\,\overset{\cup}{\cap},\ \overset{\cup}{}\!/\!\underset{\cap}{},\ \underset{\cap}{}\!\backslash\!\overset{\cup}{}\,\}$.

The left ideal generated by $\overset{\cup}{\cap}\,|$ is

$I^{(1)} = \mathrm{span}\{\overset{\cup}{\cap}\,|,\ \underset{\cap}{}\!\backslash\!\overset{\cup}{},\ \cancel{\overset{\cup}{}\!/\!\underset{\cap}{},\ |\,\overset{\cup}{\cap}}\}$ and

$I^{(2)} = \mathrm{span}\{\overset{\cup}{}\!/\!\underset{\cap}{},\ |\,\overset{\cup}{\cap}\}$ are left ideals.

Note: $I^{(1)} \cong Q$ and $I^{(2)} \cong Q$

<u>Quotients</u>

$TL_3 / \langle I^{(1)}, I^{(2)} \rangle = \mathrm{span}\{\overline{|||}\}$ with

$$\overset{\cup}{\cap}| \cdot \overline{|||} = 0 \quad \text{and} \quad |\overset{\cup}{\cap} \cdot \overline{|||} = 0.$$

If M is a ~~sub~~module and N is a submodule

N has basis $\{n_1, \ldots, n_\ell\}$

M has basis $\{n_1, \ldots, n_\ell, p_1, \ldots, p_r\}$.

Then M/N has basis $\{\bar{p}_1, \ldots, \bar{p}_r\}$

and action

$$a\bar{p}_i = \overline{ap_i} \quad \text{where } \bar{n}_1 = \cdots = \bar{n}_\ell = 0.$$

<u>The center</u>

$$Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}$$

<u>Example</u> Suppose

$$a\, \overset{\cup}{\cap}| + b\, \cap\!\backslash^{\cup} + c\, {}^{\cup}\!/\!\cap + d\, |\overset{\cup}{\cap} \in Z(A).$$

Then

$$\overset{\cup}{\cap}|\,(a(q+q^{-1}) + b) + (c(q+q^{-1}) + d)\, {}^{\cup}\!/\!\cap$$

$$= (a(q+q^{-1}) + c)\, \overset{\cup}{\cap}| + (b(q+q^{-1}) + d)\, \cap\!\backslash^{\cup}$$

So that $b = c$. and $c(q+q^{-1}) + d = 0$.

So that $b = c$ and $d = -c(q+q^{-1})$.

Similarly. $a = -b(q+q^{-1})$.

The center

$$Z(A) = \{ z \in A \mid az = za \text{ for all } a \in A \}.$$

Example Suppose

$$z = a \, \substack{\cup\\\cap}| + d \, |\substack{\cup\\\cap} + b \, \cap\backslash\cup + c \, \cup/\cap \in Z(A).$$

Then

$$\substack{\cup\\\cap}| \cdot z = (a(q+q^{-1}) + b) \substack{\cup\\\cap}| + (d + c(q+q^{-1})) \cup/\cap$$

$$= z \cdot \substack{\cup\\\cap}| = (a(q+q^{-1}) + c) \substack{\cup\\\cap}| + (d + b(q+q^{-1})) \cap\backslash\cup$$

so that $b = c$ and $d + c(q+q^{-1}) = 0$ ~~and [illegible]~~.

Also ~~$z = [illegible] \substack{\cup\\\cap}| +$~~

$$|\substack{\cup\\\cap} \cdot z = (a + c(q+q^{-1})) \cup/\cap + (d(q+q^{-1}) + b) |\substack{\cup\\\cap}$$

$$z \cdot |\substack{\cup\\\cap} = (a + b(q+q^{-1})) \cap\backslash\cup + (d(q+q^{-1}) + c) |\substack{\cup\\\cap}.$$

so that $b = c$ and $a + c(q+q^{-1}) = 0$.

So

$$z = -(q+q^{-1})(\substack{\cup\\\cap}| + |\substack{\cup\\\cap}) + (\cap\backslash\cup + \cup/\cap).$$

Then

$$z^2 = \cap\backslash\cup \left(+(q+q^{-1})^2 - (q+q^{-1})^2 - (q+q^{-1})^2 + 1 \right) + \ldots$$

$$= (1 - (q+q^{-1})^2) z.$$

So

$$z_{\emptyset} = \frac{1}{1-(q+q^{-1})^2} z \quad \text{satisfies} \quad z_{\emptyset}^2 = z_{\emptyset}.$$

$(1 - z_{\emptyset})^2 = 1 - z_{\emptyset}$ and $z_{\emptyset} + z_{\square\square} = 1$.

Semisimple algebras

An algebra is split semi~~simple~~simple if

$$A \simeq \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$$

for some index set $\hat{A}$ and some positive integers d_λ.

~~Example~~ $\bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\mathbb{C})$ has basis $\{E_{ij}^\lambda \mid \lambda \in \hat{A},\ 1 \le i,j \le d_\lambda\}$.

and multiplication

$$E_{ij}^\lambda E_{rs}^\mu = \delta_{\lambda\mu} E_{is}^\lambda \delta_{jr}.$$

Example let

$$e_{11}^\phi = \overset{\cup}{\cap}| \cdot \frac{1}{q+q^{-1}} \quad \text{and} \quad e_{22}^\phi = z_\phi - e_{11}^\phi \quad \text{and} \quad e_{11}^{\boxminus} = z_{\boxminus}$$

so

$$\left(e_{11}^\phi = \frac{1}{q+q^{-1}} \overset{\cup}{\cap}|,\ e_{22}^\phi = \frac{1}{1-(q+q^{-1})^2}\left(\cap\cup + \cup\cap\right) - \frac{(q+q^{-1})}{1-(q+q^{-1})^2}|\overset{\cup}{\cap} + \ ?\ \overset{\cup}{\cap}| \right).$$

$$e_{12}^\phi = e_{11}^\phi \frac{1}{q+q^{-1}} |\overset{\cup}{\cap}\, e_{22}^\phi, \quad e_{21}^\phi = e_{22}^\phi \frac{1}{q+q^{-1}} |\overset{\cup}{\cap}\, e_{11}^\phi.$$

Claim:

$$\begin{aligned}
A &\longrightarrow M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \\
e_{11}^\phi &\longmapsto E_{11}^\phi \\
e_{22}^\phi &\longmapsto E_{22}^\phi \\
e_{12}^\phi &\longmapsto E_{12}^\phi \\
e_{21}^\phi &\longmapsto E_{21}^\phi \\
e_{11}^{\boxminus} &\longmapsto E_{11}^{\boxminus}
\end{aligned}$$

is an isomorphism.

Here $\hat{A} = \{\phi, \boxminus\}$ and $d_\phi = 2$, $d_{\boxminus} = 1$.

Alternatively, action of TL_3 on N is

$$\substack{\cup \\ \cap}|\,(\cup\,\cdot) = (q+q^{-1})\,\cup\,\cdot \qquad \substack{\cup \\ \cap}|\,(\cdot\,\cup) = \cup\,\cdot$$

$$|\substack{\cup \\ \cap}\,(\cdot\,\cup) = (q+q^{-1})\,\cdot\,\cup \qquad |\substack{\cup \\ \cap}\,(\cup\,\cdot) = \cdot\,\cup$$

so

$$\substack{\cup \\ \cap}| \mapsto \begin{pmatrix} q+q^{-1} & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad |\substack{\cup \\ \cap} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & q+q^{-1} \end{pmatrix}$$

So

$$A \longrightarrow M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$$

$$||| \longmapsto \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\substack{\cup \\ \cap}| \longmapsto \left(\begin{array}{cc|c} q+q^{-1} & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \quad \text{is an isomorphism.}$$

$$|\substack{\cup \\ \cap} \longmapsto \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & q+q^{-1} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

Restriction and Induction

$TL_2 = \operatorname{span}\{\,||,\ \substack{\cup \\ \cap}\,\}$ with $(q+q^{-1})\substack{\cup \\ \cap} = \substack{\cup \\ \cap}\cdot\substack{\cup \\ \cap}$.

Simple modules: $\mathbb{C}v_\phi$ and $\mathbb{C}v_{\Box\Box}$

$$\substack{\cup \\ \cap}\,v_\phi = (q+q^{-1})v_\phi \quad \text{and} \quad \substack{\cup \\ \cap}\,v_{\Box\Box} = 0$$

Then $z_\phi = e_{11}^\phi = \frac{1}{q+q^{-1}}\substack{\cup \\ \cap}$ and $z_{\Box\Box} = 1 - z_\phi$.

 $\longmapsto \left(\begin{array}{cc|c} q+q^{-1} & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$  $\longmapsto \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & q+q^{-1} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$

Then

 $\longmapsto \left(\begin{array}{cc|c} 1 & q+q^{-1} & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$  $\longmapsto \left(\begin{array}{cc|c} 0 & 0 & 0 \\ q+q^{-1} & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$

So

 $\longmapsto \begin{pmatrix} 1 & q+q^{-1} \\ q+q^{-1} & 1 \end{pmatrix}$  $= \begin{pmatrix} q+q^{-1} & 1 \\ 1 & q+q^{-1} \end{pmatrix}$

 $= \begin{pmatrix} 1-(q+q^{-1})^2 & 0 \\ 0 & 1-(q+q^{-1})^2 \end{pmatrix}$