The algebra $A = M_d(\mathbb{C})$

Let $\mathbb{C}^d = \mathrm{span}\{e_i \mid 1 \le i \le d\}$ where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

with $M_d(\mathbb{C})$-action by left multiplication

Theorem

(1) If M is a simple $M_d(\mathbb{C})$-module then $M \cong \mathbb{C}^d$

(2) If $t: M_d(\mathbb{C}) \to \mathbb{C}$ is a trace then $t = \kappa \cdot Tr$, with $\kappa \in \mathbb{C}$.

(3) $Z(M_d(\mathbb{C})) = \mathbb{C} \cdot Id$

(4) $M_d(\mathbb{C})$ has one nonzero ideal.

Proof (1) Let M be a simple $M_d(\mathbb{C})$-module. Let $m \in M$ be nonzero. Since

$$m = 1 \cdot m = \sum_{i=1}^{d} E_{ii} m, \quad \text{then } E_{ii} m \neq 0 \text{ for some } i.$$

Let $m_j = E_{ji} m$ for $j = 1, 2, \ldots, d$. Then

$N = \mathrm{span}\{m_1, \ldots, m_d\}$ is a submodule of M such that $E_{rs} m_j = \delta_{sj} m_r$.

Since M is simple, $N = M$ and

$$\begin{aligned} M &\longrightarrow \mathbb{C}^d \\ m_j &\longmapsto e_j \end{aligned}$$ ~~is and $M_d(\mathbb{C})$~~

is an $M_d(\mathbb{C})$-module isomorphism.

(2) Let $t: M_d(\mathbb{C}) \to \mathbb{C}$ be a trace. Then

$$t(E_{ij}) = t(E_{ii}E_{ij}) = t(E_{ij}E_{ii}) = \delta_{ij}\, t(E_{11}).$$

So $t = t(E_{11}) \cdot \mathrm{Tr}$. //.

The algebra $A = \bigoplus_{\lambda\in\hat{A}} M_{d_\lambda}(\mathbb{C})$

A has basis $\{E_{ij}^\lambda \mid 1 \le i,j \le d_\lambda,\ \lambda\in\hat{A}\}$ with

$$E_{ij}^\lambda E_{rs}^\mu = \delta_{\lambda\mu}\delta_{jr} E_{is}^\lambda.$$

Define $A^\lambda = \mathrm{span}\{e_i^\lambda \mid 1 \le i \le d_\lambda\}$ with

$$E_{ij}^\mu e_r^\lambda = \delta_{\lambda\mu}\delta_{jr} e_i^\lambda.$$

Define $\mathrm{Tr}^\lambda : A \to \mathbb{C}$ by $\mathrm{Tr}^\lambda(E_{ij}^\mu) = \delta_{\lambda\mu}\delta_{ij}$.

Define $z_\lambda = \sum_{i=1}^{d} E_{ii}^\lambda$, so that $z_\lambda^2 = z_\lambda$ and $1 = \sum_{\lambda\in\hat{A}} z_\lambda$.

Define $I^\lambda = z_\lambda A = \mathrm{span}\{E_{ij}^\lambda \mid 1 \le i,j \le d_\lambda\}$.

<u>Theorem</u> (1) A^λ, $\lambda\in\hat{A}$, are the simple A-modules.

(2) If $t: A \to \mathbb{C}$ is a trace then

$$t = \sum_{\lambda\in\hat{A}} t_\lambda \mathrm{Tr}^\lambda, \quad \text{with } t_\lambda \in \mathbb{C}.$$

(3) $Z(A) = \mathrm{span}\{z_\lambda \mid \lambda\in\hat{A}\}$

(4) The minimal ideals of A are I^λ, $\lambda\in\hat{A}$.
The ideals of A are sums of I^λ.

The regular representation of A

$$A = \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus d_\lambda} \quad \text{since} \quad A = \left\{ \begin{pmatrix} \boxed{} & & & 0 \\ & \boxed{} & & \\ & & \boxed{} & \\ 0 & & & \ddots \end{pmatrix} \right\}$$

Note that

$$\begin{aligned} Tr^\lambda : A &\to \mathbb{C} \\ a &\mapsto Tr(\rho^\lambda(a)) \end{aligned} \qquad \text{where} \qquad \begin{aligned} \rho^\lambda : A &\to M_{d_\lambda}(\mathbb{C}) \\ a &\mapsto \rho^\lambda(a), \end{aligned}$$

where $\rho^\lambda(a)$ is the matrix of the action of a on A^λ.
If $t : A \to \mathbb{C}$ is the trace of the regular representation then

$$t = \sum_{\lambda \in \hat{A}} d_\lambda Tr^\lambda .$$

The dual basis to $\{E_{ij}^\lambda \mid \lambda \in \hat{A},\ 1 \le i,j \le d_\lambda\}$ is

$$\left\{\frac{1}{d_\lambda} E_{ji}^\lambda \;\middle|\; \lambda \in \hat{A},\ 1 \le i,j \le d_\lambda\right\}$$

since $t\left(E_{ij}^\lambda \frac{1}{d_\mu} E_{sr}^\mu\right) = \delta_{ir}\delta_{\lambda\mu}\delta_{js}$. Thus

$$E_{ij}^\lambda = \sum_{\substack{\mu \in \hat{A} \\ 1 \le r,s \le d_\mu}} d_\mu A^\mu\left(\frac{1}{d_\mu} E_{sr}^\mu\right)_{ji} E_{rs}^\mu .$$

If $B = \{b\}$ is a basis of A and $\{b^*\}$ is the dual basis with respect to $\langle , \rangle$ defined by t.
Then

$$E_{ij}^\lambda = \sum_{b \in B} d_\lambda A^\lambda(b^*)_{ji}\, b \quad \text{and} \quad b = \sum_{\substack{\lambda \in \hat{A} \\ 1 \le i,j \le d_\lambda}} A^\lambda(b)_{ij} E_{ij}^\lambda$$

Theorem (Artin-Wedderburn)

Let A be a finite dimensional algebra ~~with~~ such that the trace of the regular representation is nondegenerate. Then

$$A \xrightarrow{\sim} \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}) \quad \text{as algebras}$$

where $A = \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus d_\lambda}$ as A-modules.

Proof By Maschke's theorem

$$A \cong \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus d_\lambda} \quad \text{as } A\text{-modules.}$$

Since $\begin{matrix} A \to \mathrm{End}(A) \\ a \mapsto a_A \end{matrix}$ is injective, $\begin{matrix} \rho : A \to \mathrm{End}\left(\bigoplus_{\lambda \in \hat{A}} A^\lambda\right) \\ a \mapsto \bigoplus_{\lambda \in \hat{A}} \rho^\lambda(a) \end{matrix}$

is also injective and ρ is an algebra homomorphism.

<u>Example</u> Temperley-Lieb $TL_3 = \mathrm{span}\{|||, \overset{\cup}{\cap}|, |\overset{\cup}{\cap}, \cup/\cap, \cap\backslash\cup\}$ has module $M = \mathrm{span}\{\cup\cdot, \cdot\cup, \dot{\cup}\}$ which we found was

$$M = N \oplus P \quad \text{where} \quad N = \mathrm{span}\{\cdot\cup, \cup\cdot\} \text{ and}$$
$$P = \mathrm{span}\{(1+(q+q^{-1}))\dot{\cup} - \cup\cdot - \cdot\cup\}$$

We have

$$\rho^{\boxplus} : TL_3 \longrightarrow M_1(\mathbb{C}) \quad \text{and} \quad \rho^{\Phi} : TL_3 \longrightarrow M_2(\mathbb{C})$$
$$\overset{\cup}{\cap}| \longmapsto 0 \qquad\qquad \overset{\cup}{\cap}| \longmapsto \begin{pmatrix} q+q^{-1} & 1 \\ 0 & 0 \end{pmatrix}$$
$$|\overset{\cup}{\cap} \longmapsto 0 \qquad\qquad |\overset{\cup}{\cap} \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & q+q^{-1} \end{pmatrix}.$$

Then

$$TL_3 \longrightarrow \bigoplus_{\lambda \in \hat{TL}_3} M_{d_\lambda}(\mathbb{C})$$
$$\overset{\cup}{\cap}| \longmapsto \left(\begin{array}{cc|c} q+q^{-1} & 1 & 0 \\ 0 & 0 & \\ \hline 0 & & 0 \end{array}\right)$$
$$|\overset{\cup}{\cap} \longmapsto \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & q+q^{-1} & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

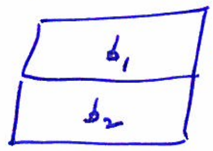
Let G be a group. The <u>group algebra</u> of G is the algebra $\mathbb{C}G$ with basis G and multiplication determined by the multiplication in G. The map

$$t : \mathbb{C}G \longrightarrow \mathbb{C} \quad \text{given by} \quad t(a) = a|_1,$$

the coefficient of 1 in a, is a trace on G. If Tr is the trace of the regular representation of G then

$$Tr(g) = \sum_{h \in G} gh|_h = \begin{cases} |G|, & \text{if } g = 1 \\ 0, & \text{if } g \neq 1 \end{cases} = |G| \cdot t(g).$$

The *braid group* B_n is the group of braids with n strands with product

$b_1 b_2 =$ 

Theorem (Artin) B_n is presented by generators $g_1, \ldots, g_{n-1}$, where

$$g_i = \overset{1\ 2\ \cdots\ i\ i+1\ \cdots\ n}{||||\times|||||},$$

with relations $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$.

The *symmetric group* S_n is the quotient of B_n by the relations $g_i^2 = 1$.

(write $s_i = ||||\overset{i\ i+1}{\times}|||$ since $g_i = ||||\times|||||$
$= g_i^{-1} = ||||\times||||$ in S_n)

The *Iwahori-Hecke algebra* is the quotient of $\mathbb{C}B_n$ by the relations

$$(T_i - q)(T_i + q^{-1}) = 0. \qquad (T_i^2 = (q - q^{-1})T_i + 1).$$

Let

$$e_i = q - T_i \quad \text{in } H_n.$$

Then $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ becomes

$$e_i e_{i+1} e_i - e_{i+1} e_i e_{i+1} = e_i - e_{i+1}.$$

The map

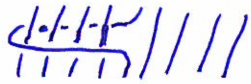
$\mathbb{C}B_n \longrightarrow H_n \longrightarrow TL_n$ ~~is a~~ are surjective algebra

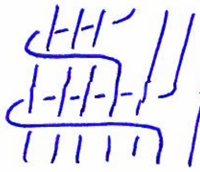
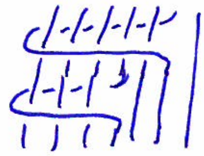
$e_i \longmapsto e_i$

$T_i \longmapsto T_i$

homomorphisms

<u>Remark</u> $T_{w_0}^2 =$  $\in Z(B_n)$

and $T_{w_0}^2 = y^{\varepsilon_1} \cdots y^{\varepsilon_n}$, where $y^{\varepsilon_i} =$ 

and $y^{\varepsilon_i} y^{\varepsilon_j} =$  $=$  $= y^{\varepsilon_j} y^{\varepsilon_i}$, for $1 \le i, j \le n$.

So the image of $\mathbb{C}[y^{\varepsilon_1}, \ldots, y^{\varepsilon_n}]$ is a large commutative subalgebra of H_n (or TL_n).

Note:

$$B_n \hookrightarrow B_{n+1}$$

$$b \longmapsto \boxed{b}\,|$$ gives inclusions

$H_1 \subseteq H_2 \subseteq H_2 \subseteq \cdots,$

$TL_1 \subseteq TL_2 \subseteq TL_3 \subseteq \cdots,$ and $\mathbb{C}S_1 \subseteq \mathbb{C}S_2 \subseteq \mathbb{C}S_3 \subseteq \cdots$.

<u>Pullback functors</u> Suppose $A \xrightarrow{\varphi} R$ is an algebra homomorphism. Then we get a functor

$$\{R\text{-modules}\} \longrightarrow \{A\text{-modules}\}$$

$$M \longmapsto \varphi^*(M)$$

where $\varphi^*(M) = M$, as vector spaces and the A-action is given by

$$a \cdot m = \varphi(a) m, \text{ for } a \in A,\ m \in M.$$

The map $H_n \xrightarrow{\pi} TL_n$ gives a functor

$$\{TL_n\text{-modules}\} \longrightarrow \{H_n\text{-modules}\}$$

which takes simple modules to simple modules.
The map π is surjective and the map π^* is injective.
The map $TL_3 \overset{\iota}{\hookrightarrow} TL_4$ gives $\{TL_4\text{-modules}\} \xrightarrow{\iota^*} \{TL_3\text{-modules}\}$
The functor ι^* is Restriction, $\mathrm{Res}^{TL_4}_{TL_3}$.

<u>Adjoint functors</u>

Let $F: \{A\text{-modules}\} \longrightarrow \{B\text{-modules}\}$ be a functor.
The <u>adjoint functor</u> $F^\vee: \{B\text{-modules}\} \longrightarrow \{A\text{-modules}\}$ is determined by

$$\mathrm{Hom}_{B\text{-mod}}(F^\vee M, N) \simeq \mathrm{Hom}_{A\text{-mod}}(M, FM).$$

The adjoint functor to Res^B_A is <u>induction</u> Ind^B_A.

$$\mathrm{Ind}^B_A : \{A\text{-modules}\} \longrightarrow \{B\text{modules}\}.$$

It is given explicitly by

$$\mathrm{Ind}^B_A(M) = B \otimes_A M,$$

where $B \otimes_A M$ is generated by $b \otimes m$, $b \in B$, $m \in M$, with relations

$$ba \otimes m = b \otimes am$$

and bilinearity.

Bratelli diagrams

Let
$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$
be a sequence of inclusions of semisimple algebras. The Bratelli diagram for $A_1 \subseteq A_2 \subseteq \cdots$ is the graph with

vertices on level k: $\hat{A}_k$,

where $\hat{A}_k$ is an index set for the simple A_k-modules and

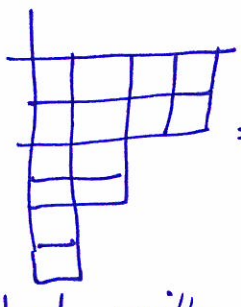
$m_{\lambda\mu}$ edges connecting λ and μ $(\lambda \in \hat{A}_k, \mu \in \hat{A}_{k-1})$

if
$$\mathrm{Res}^{A_k}_{A_{k-1}}(A_k^\lambda) = \bigoplus_{\mu \in \hat{A}_{k-1}} (A_{k-1}^\mu)^{\oplus m_{\lambda\mu}}.$$

Theorem The Bratelli diagram for
$$TL_1 \subseteq TL_2 \subseteq \cdots$$
has vertices on level k: $\left\{ \begin{matrix} \text{partitions of } k \\ \text{with } \le 2 \text{ rows} \end{matrix} \right\} = \hat{TL}_k$

and an edge $\lambda - \mu$ if μ is obtained from λ by removing a box.

A partition is a collection of boxes in a corner.

$\lambda =$  $= (442211)$

Write $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 \ge \cdots \ge \lambda_\ell$ and $\lambda_i = \#$ of boxes in row i.

<u>Theorem</u> (a) The Bratelli diagram for
$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$$
has
$$\hat{H}_k = \left\{ \begin{matrix} \text{partitions} \\ \text{with } k \text{ boxes} \end{matrix} \right\} \quad \text{and} \quad \lambda - \mu$$
if μ is obtained from λ by removing a box.

(b) The Bratelli diagram for
$$\mathbb{C}S_1 \subseteq \mathbb{C}S_2 \subseteq \mathbb{C}S_3 \subseteq \cdots$$
has
$$\hat{S}_k = \left\{ \begin{matrix} \text{partitions} \\ \text{with } k \text{ boxes} \end{matrix} \right\} \quad \text{and} \quad \lambda - \mu$$
if μ is obtained from λ by removing a box.

The algebra $U\mathfrak{sl}_2$

A <u>Lie algebra</u> is a vector space $\mathfrak{g}$ with a bracket $[\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

(1) $[x,y] = -[y,x]$, for $x, y \in \mathfrak{g}$,

(2) $[[x,y],z] + [[z,x],y] + [[y,z],x] = 0$, for $x, y, z \in \mathfrak{g}$.

A Lie algebra is not an algebra.

The <u>enveloping</u> <u>algebra</u> of $\mathfrak{g}$ is the algebra $U\mathfrak{g}$ generated by the vector space $\mathfrak{g}$, with the relations
$$xy = yx + [x,y], \quad \text{for } x, y \in \mathfrak{g}.$$

The <u>Lie algebra $\mathfrak{sl}_2$</u> is the vector space

$$\mathfrak{sl}_2 = \{ a \in M_2(\mathbb{C}) \mid \operatorname{tr} a = 0 \}$$

with bracket

$$[a,b] = ab - ba, \quad \text{for } a,b \in \mathfrak{sl}_2$$

(where the product on the RHS is matrix multiplication).

The enveloping algebra of $\mathfrak{sl}_2$ is the algebra $U\mathfrak{sl}_2$ generated by x, y, k with relations

$$xy = yx + k, \qquad kx = xk + 2x, \qquad ky = yk - 2y.$$

<u>Proposition</u> The Lie algebra $\mathfrak{sl}_2$ is presented by generators

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and relations

$$[x,y] = k, \quad [k,x] = 2x, \quad [k,y] = -2y.$$

<u>HW</u>: Show that $U\mathfrak{sl}_2$ has basis

$$\{ y^{m_1} k^{m_2} x^{m_3} \mid m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0} \}.$$

Hence $\dim(U\mathfrak{sl}_2) = 3\infty$.