REPRESENTATION THEORY

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Abstract. Notes from Arun Ram's 2008 course at the University of Melbourne.

5. Week 5

Definition. $\mathfrak{sl}_2$ is the Lie algebra consisting of matrices

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a + d = 0\}$$

with brackett

$$[x, y] = xy - yx.$$

$\mathfrak{sl}_2$ is presented by e, f, h with relations

$$[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f,$$

where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Definition. $\mathcal{U}(\mathfrak{sl}_2)$ is the algebra generated by e, f, h with relations

$$ef = fe + h,\ eh = he - 2e,\ hf = fh - 2f.$$

So $\mathcal{U}(\mathfrak{sl}_2)$ has basis $\{f^{m_1} h^{m_2} e^{m_3} | m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}\}$.

Date: August 27, 2008.
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If $M = \text{span}\{m_1, \ldots, m_r\}$ and $N = \text{span}\{n_1, \ldots, n_s\}$ are $\mathcal{U}(\mathfrak{sl}_2)$-modules then $M \otimes N = \text{span}\{m_i \otimes n_j\}$ has $\mathcal{U}(\mathfrak{sl}_2)$-action given by

$$\begin{aligned} e(m_i \otimes n_j) &= em_i \otimes n_j + m_i \otimes en_j \\ f(m_i \otimes n_j) &= fm_i \otimes n_j + m_i \otimes fn_j \\ h(m_i \otimes n_j) &= hm_i \otimes n_j + m_i \otimes hn_j. \end{aligned}$$

$\mathcal{U}_q(\mathfrak{sl}_2)$ is an algebra and a specialization of $\mathcal{U}(\mathfrak{sl}_2)$:

$$\mathcal{U}_q(\mathfrak{sl}_2) \xrightarrow{q=1} \mathcal{U}(\mathfrak{sl}_2)$$

Definition. $\mathcal{U}_q(\mathfrak{sl}_2)$ has generators E, F, $K^{\pm 1}$ and relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 \\ KEK^{-1} &= q^2 E \\ KFK^{-1} &= q^{-2} F \\ EF &= FE + \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

Note: $KE = q^2 EK$, $KF = q^{-2} FK$, and $\mathcal{U}_q(\mathfrak{sl}_2)$ has basis

$$\{F^{m_1} K^{m_2} E^{m_3} | m_1, m_3 \in \mathbb{Z}_{\geq 0}, m_2 \in \mathbb{Z}\}$$

$\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $M \otimes N$ by[1]

$$\begin{aligned} E(m_i \otimes n_j) &= Em_i \otimes Kn_j + m_i \otimes En_j \\ F(m_i \otimes n_j) &= Fm_i \otimes n_j + K^{-1}m_i \otimes Fn_j \\ K(m_i \otimes n_j) &= Km_i \otimes Kn_j \end{aligned}$$

5.1. $\mathcal{U}_q(\mathfrak{sl}_2)$**-modules.** Our building block here is the two-dimensional simple $\mathcal{U}_q(\mathfrak{sl}_2)$-module $L(\Box)$.

Definition. $L(\Box) = \text{span}\{v_1, v_{-1}\}$ with $\mathcal{U}_q(\mathfrak{sl}_2)$-action

$$\begin{array}{lll} Ev_1 = 0, & Fv_1 = v_{-1}, & Kv_1 = qv_1 \\ Ev_{-1} = v_1, & Fv_{-1} = 0, & Kv_{-1} = q^{-1}v_{-1} \end{array}$$

[1]Note that this makes use of a slightly different coproduct Δ than the one we defined last week.

In this basis, E, F and K act as

$$\rho^{\square}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho^{\square}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho^{\square}(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

We'll build up more modules from this one by tensoring: $L(\square) \otimes L(\square)$ has basis $\{v_1 \otimes v_1, v_{-1} \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}$.

Let's figure out how F acts here.

$$\begin{array}{c} v_1 \otimes v_1 \\ \downarrow F \\ v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1} \\ \downarrow F \\ 0 + q v_{-1} \otimes v_{-1} + q^{-1} v_{-1} \otimes v_{-1} + q^{-2} 0 = [2] v_{-1} \otimes v_{-1} \\ \downarrow F \\ 0 \end{array}$$

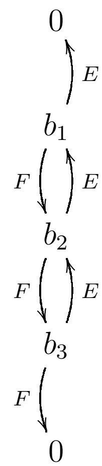
Let $b_1 = v_1 \otimes v_1$, $b_2 = v_{-1} \otimes v_1 + q^{-1} v_1 \otimes v_{-1}$, $b_3 = v_{-1} \otimes v_{-1}$, $b_4 = v_{-1} \otimes v_1 - q v_1 \otimes v_{-1}$. Now we calculate $Eb_4 = q v_1 \otimes v_1 + 0 - q0 - q v_1 \otimes v_1 = 0$, $Fb_4 = 0$ and $Kb_4 = q^{-1} q b_4 = b_4$. So $L(\emptyset) = \operatorname{span}\{b_4\}$ is a submodule of $L(\square) \otimes L(\square)$.

$L(\square\square) = \operatorname{span}\{b_1, b_2, b_3\}$ is another $\mathcal{U}_q(\mathfrak{sl}_2)$-submodule of $L(\square) \otimes L(\square)$ You can compute the action of each of E, F and K on the basis $\{b_1, b_2, b_3\}$ and see

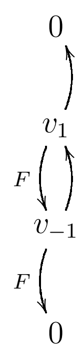
$$\rho^{\square\square}(F) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & [2] & 0 \end{pmatrix}, \quad \rho^{\square\square}(E) = \begin{pmatrix} 0 & [2] & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\rho^{\square\square}(K) = \begin{pmatrix} q^2 & 0 & 0 \\ 0 & q^0 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix}$$

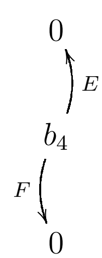
Up to constants, we picture the action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $L(\square\square)$ as



and on $L(\square)$ as



and on $L(\emptyset)$ at



So we've seen $L(\square) \otimes L(\square) = L(\square\square) \oplus L(\emptyset)$.

Now

$$\begin{aligned} L(\square)\otimes L(\square)\otimes L(\square) =& (L(\square\square)\oplus L(\emptyset))\otimes L(\square) \\ =& (L(\square\square)\otimes L(\square))\oplus(L(\square)\otimes L(\emptyset)) \\ =& \mathrm{span}\{b_1\otimes v_1, b_2\otimes v_1, b_3\otimes v_1, b_4\otimes v_1, \\ & b_1\otimes v_{-1}, b_2\otimes v_{-1}, b_3\otimes v_{-1}, b_4\otimes v_{-1}\}, \end{aligned}$$

and we calculate

$$\begin{array}{c} b_1\otimes v_1 \\ {\scriptstyle F}\downarrow \\ b_2\otimes v_1 + q^{-2}b_1\otimes v_{-1} \\ {\scriptstyle F}\downarrow \\ [2]b_3\otimes v_1 + q^{-1}[2]b_2\otimes v_{-1} \\ {\scriptstyle F}\downarrow \\ [2][3]b_3\otimes v_{-1} \\ {\scriptstyle F}\downarrow \\ 0 \end{array}$$

Letting $c_1 = b_1\otimes v_1$, $c_2 = b_2\otimes v_1 + q^{-2}b_1\otimes v_{-1}$, $c_3 = b_3\otimes v_1 + q^{-1}b_2\otimes v_{-1}$, and $c_4 = b_3\otimes v_{-1}$, we have (up to constants)

$$\begin{array}{c} 0 \\ \quad\uparrow{\scriptstyle E} \\ c_1 \\ {\scriptstyle F}\downarrow\uparrow{\scriptstyle E} \\ c_2 \\ {\scriptstyle F}\downarrow\uparrow{\scriptstyle E} \\ c_3 \\ {\scriptstyle F}\downarrow\uparrow{\scriptstyle E} \\ c_4 \\ {\scriptstyle F}\downarrow\quad \\ 0 \end{array}$$

and

$$\rho^{\square\square\square}(F) = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & [2] & 0 & \\ & & [3] & 0 \end{pmatrix}, \quad \rho^{\square\square\square}(E) = \begin{pmatrix} 0 & [3] & & \\ & 0 & [2] & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix},$$

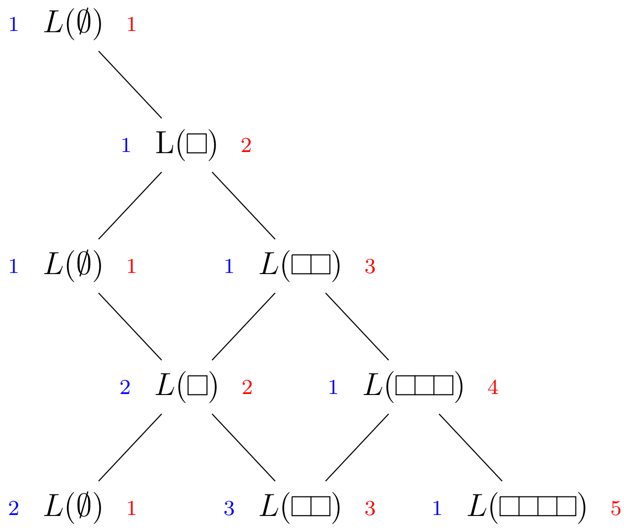
$$\rho^{\square\square\square}(K) = \begin{pmatrix} q^{-3} & & & \\ & q^{-1} & & \\ & & q & \\ & & & q^3 \end{pmatrix}$$

Letting $L(\square\square\square) = \operatorname{span}\{c_1, c_2, c_3, c_4\}$, we pick up our previous calculation:

$$\begin{aligned} L(\square) \otimes L(\square) \otimes L(\square) &= (L(\square\square) \oplus L(\emptyset)) \otimes L(\square) \\ &= (L(\square\square) \otimes L(\square)) \oplus (L(\square) \otimes L(\emptyset)) \\ &= L(\square\square\square) \oplus L(\square) \oplus L(\square) \end{aligned}$$

At least by counting dimensions, this seems true. The first $L(\square)$ is $\operatorname{span}\{c_5, c_6\}$ for $c_5 = b_2 \otimes v_1 - q b_1 \otimes v_{-1}$, $c_6 = [2] b_3 \otimes v_{-1} - q^2 b_2 \otimes v_{-1}$. The second $L(\square)$ is $\operatorname{span}\{c_7, c_8\}$ – you work out what c_7 and c_8 are, and then compute that these really are irreducible modules isomorphic to $L(\square)$.

At this point, we've seen enough of how this process works that we can build a Bratelli diagram. We'll put the dimensions of each module in red, and the number of times it appears in $L(\square)^{\otimes k}$ in blue.



5.2. **Temperley-Lieb, and Schur-Weyl duality.** Now of course, if we add up the product of the red and blue numbers across the kth row, we get 2^k, the dimension of $L(\square)^{\otimes k}$. But you might also notice that if we add up the squares of the blue numbers across each row, we get the Catalan numbers – the dimensions of TL_k.

This means we have a theorem coming up. We don't know what the theorem says because we don't know how to prove it yet, but the numerology here suggests there's some connection between Temperley-Lieb and $\mathcal{U}_q(\mathfrak{sl}_2)$.

We will abreviate $L(\square)$ to V when convenient. Define an action of TL_2 on

$$V^{\otimes 2} = L(\square) \otimes L(\square) = \text{span}\{v_1 \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_1, v_{-1} \otimes v_{-1}\}$$

by

$$\substack{\smile \\ \frown}(v_1 \otimes v_1) = 0 \qquad \substack{\smile \\ \frown}(v_1 \otimes v_{-1}) = q v_1 \otimes v_{-1} - v_{-1} \otimes v_1$$
$$\substack{\smile \\ \frown}(v_{-1} \otimes v_{-1}) = 0 \qquad \substack{\smile \\ \frown}(v_{-1} \otimes v_1) = q^{-1} v_{-1} \otimes v_1 - v_1 \otimes v_{-1}$$

Of course we need to verify that this really is an action of Temperley-Lieb, so we must check some relations, such as

$$\begin{aligned}\underset{\frown}{\smile}\left(\underset{\frown}{\smile}(v_1\otimes v_{-1})\right) &= \underset{\frown}{\smile}(qv_1\otimes v_{-1}-v_{-1}\otimes v_1)\\ &= q(qv_1\otimes v_{-1}-v_{-1}\otimes v_1)-q^{-1}v_{-1}\otimes v_1 - v_1\otimes v_{-1}\\ &= [2](qv_1\otimes v_{-1}-v_{-1}\otimes v_1) = [2]\underset{\frown}{\smile}(v_1\otimes v_{-1})\\ &= \left(\underset{\frown}{\smile}\right)^2(v_1\otimes v_{-1})\end{aligned}$$

Recall $b_4 = v_{-1}\otimes v_1 - qv_1\otimes v_{-1}$. In fact $\frac{1}{[2]}\underset{\frown}{\smile}$ is a projection onto $L(\emptyset)$ inside $L(\square)\otimes L(\square)$ and the action of TL_2 on $V^{\otimes 2}$ commutes with the $\mathcal{U}_q(\mathfrak{sl}_2)$ action on $V^{\otimes 2}$, ie $E\cdot\underset{\frown}{\smile} = \underset{\frown}{\smile}\cdot E$, $F\cdot\underset{\frown}{\smile} = \underset{\frown}{\smile}\cdot F$, $K\cdot\underset{\frown}{\smile} = \underset{\frown}{\smile}\cdot K$ on $V^{\otimes 2}$.

Recall. TL_k is generated by $e_j = \left|\cdots\right|\underset{\frown}{\smile}\left|\cdots\right|$, with relations $e_i^2 = [2]e_i$ and $e_ie_{i\pm1}e_1 = e_i$

Definition. We define an action of TL_k on

$$V^{\otimes k} = L(\square)\otimes L(\square)\otimes\cdots\otimes L(\square) = \operatorname{span}\left\{v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}|i_1,\ldots,i_k\in\{1,-1\}\right\},$$

a 2^k-dimensional module, by letting

$$e_j(v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_k}) = v_{i_1}\otimes\cdots\otimes v_{i_{j-1}}\otimes\underset{\frown}{\smile}(v_{i_j}\otimes v_{i_{j+1}})\otimes v_{i_{j+2}}\otimes\cdots\otimes v_{i_k}.$$

You should check that this really is an action of TL_k.

This TL_k-action on $V^{\otimes k}$ commutes with the $\mathcal{U}_q(\mathfrak{sl}_2)$-action. Why is this good? Let A be an algebra and let M be a semisimple A-module, so

$$M = \bigoplus_{\lambda\in\hat{M}}(A^\lambda)^{\oplus m_\lambda}.$$

Consider the centralizer algebra of M, ie let $Z = \mathrm{End}_A(M) = \{z\in\mathrm{End}(M)|za = az \text{ for all } a\in A\}$. (So $Z = TL_k$ in this particular example).

Theorem 5.1. *$Z = \bigoplus_{\lambda \in \hat{M}} M_{m_\lambda}(\mathbb{C})$, which has irreducible Z-modules Z^λ. As an (A, Z) bimodule (or an $A \otimes Z$ module),*

$$M \simeq \bigoplus_{\lambda \in \hat{M}} A^\lambda \otimes Z^\lambda.$$

The above is sometimes called Schur-Weyl duality (A centralizer pair commuting with each other on the same module.)

So why is this true?

Proof. By definition

$$\begin{aligned} Z &= \mathrm{End}_A(M) = \mathrm{Hom}_A(M, M) \\ &= \mathrm{Hom}_A\Big(\bigoplus_{\lambda} \bigoplus_{i-1}^{m_\lambda} A_i^\lambda, \bigoplus_{\lambda} \bigoplus_{i-1}^{m_\lambda} A_i^\lambda\Big) \\ &= \bigoplus_{\mu,\lambda} \bigoplus_{i=1}^{m_\lambda} \bigoplus_{j=1}^{m_\lambda} \mathrm{Hom}_A(A_i^\lambda, A_i^\mu) \end{aligned}$$

but by Schur's lemma,

$$= \bigoplus_{\lambda \in \hat{M}} \bigoplus_{i,j=1}^{m_\lambda} \mathrm{Hom}_A(A_i^\lambda, A_i^\lambda)$$

up to constants, $e_{i,j}^\lambda : A_i^\lambda \to A_j^\lambda$ is the unique element of $\mathrm{Hom}_A(A_i^\lambda, A_j^\lambda)$, so

$$= \bigoplus_{\lambda \in \hat{M}} \bigoplus_{i,j=1}^{m_\lambda} \mathbb{C} e_{i,j}^\lambda = \bigoplus_{\lambda \in \hat{M}} M_{m_\lambda}(\mathbb{C})$$

□

One example of Schur-Weyl duality is what we just saw, where $\mathcal{U}_q(\mathfrak{sl}_2)$ commutes with TL_k on $V^{\otimes k}$.

Another example is given by GL_n and S_k. Let $GL_n = \{g \in M_n | g \text{ is invertible}\}$. $V = \mathrm{span}\,\{v_1, \ldots, v_n\}$ with $gv_i = \sum_{j=1}^n g_{j,i} v_j$ and GL_n acts on $V^{\otimes k}$ by $g(v_{i_1} \otimes \cdots \otimes v_{i_k}) = gv_{i_1} \otimes \cdots \otimes gv_{i_k}$. We also have the action of S_k on $V^{\otimes k}$ by permuting the tensor entries.

The S_k action commutes with the GL_n-action. (This is the classical case of Schur-Weyl duality.) After this week's homework you know everything about S_k representations, so you should be able to figure out everything about GL_n representations.

Question. Did we prove the second statement in the theorem?

Answer. Z acts on M, $1 = \sum_{\lambda \in \hat{M}} \sum_{i=1}^{m_\lambda} e_{i,i}^\lambda$ so

$$\begin{aligned} M &= \sum_\lambda \sum_i e_{i,i}^\lambda M \\ &= \sum_\lambda \sum_i A_i^\lambda \\ &= \sum_\lambda \sum_i e_{1,i}^\lambda M \end{aligned}$$

and because span $\left\{ e_{1,i}^\lambda | 1 \leq i \leq m_\lambda \right\} = Z^\lambda$,

$$= \sum_\lambda Z^\lambda \otimes A^\lambda$$

Next week, R. Brak will be lecturing about crystals.