$\mathfrak{sl}_2$ Crystals

Start with

$B(\square) = \{\rightarrow, \leftarrow\}$ with operators $\tilde{e}$ and $\tilde{f}$

given by

$$\tilde{e}(\leftarrow) = \rightarrow, \qquad \tilde{f}(\leftarrow) = 0$$
$$\tilde{e}(\rightarrow) = 0, \qquad \tilde{f}(\rightarrow) = \leftarrow.$$

$$\begin{tikzcd} \rightarrow \arrow[d, bend right, "\tilde{f}"'] \\ \leftarrow \arrow[u, bend right, "\tilde{e}"'] \end{tikzcd}$$

<u>Tensor products</u> are by concatenation

$$B(\square)\otimes B(\square) = \{\rightarrow\rightarrow,\ \rightarrow\leftarrow,\ \leftarrow\rightarrow,\ \leftarrow\leftarrow\}$$

$$B(\square)\otimes B(\square)\otimes B(\square) = \left\{\begin{array}{l} \rightarrow\rightarrow\rightarrow,\ \rightarrow\leftarrow\rightarrow,\ \leftarrow\rightarrow\rightarrow,\ \leftarrow\rightarrow\leftarrow \\ \rightarrow\rightarrow\leftarrow,\ \rightarrow\leftarrow\leftarrow,\ \leftarrow\leftarrow\rightarrow,\ \leftarrow\leftarrow\leftarrow \end{array}\right\}$$

If B_1 and B_2 are $\mathfrak{sl}_2$-crystals then

$\tilde{f}$ acts on $B_1 \otimes B_2 = \{p\otimes q \mid p \in B_1, q \in B_2\}$

by

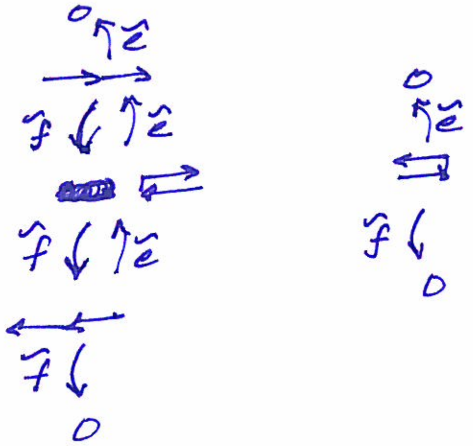
$$\tilde{f}(p\otimes q) = \begin{cases} \tilde{f}p\otimes q, & \text{if (last occurrence of) most negative point of } p\otimes q \text{ is in } p, \\ p\otimes \tilde{f}q, & \text{if (last occurrence of) most negative point of } p\otimes q \text{ is in } q \end{cases}$$

and the action of $\tilde{e}$ is given by

$$\tilde{e}b = \begin{cases} b', & \text{if } b = \tilde{f}b', \\ 0, & \text{otherwise.} \end{cases}$$

Decomposing $B^{\otimes k}$ where $B = B(\square)$.

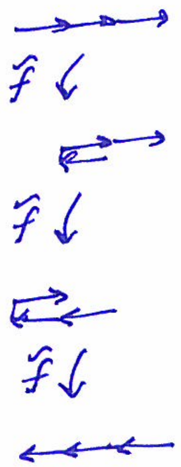
$B(\square)\otimes B(\square) = \{\rightarrow\rightarrow,\ \leftrightarrows,\ \rightleftarrows,\ \leftarrow\leftarrow\}$ with

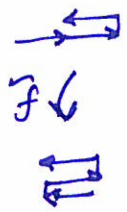


So $B(\square)\otimes B(\square) = B(\square\square) \sqcup B(\phi)$, where

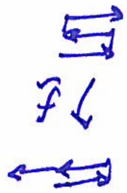
$B(\square\square) = \{\rightarrow\rightarrow,\ \rightleftarrows,\ \leftarrow\leftarrow\}$ and $B(\phi) = \{\leftrightarrows\}$.

Then
$$\begin{aligned} B^{\otimes 3} &= \big(B(\square)\otimes B(\square)\big)\otimes B(\square) \\ &= \big(B(\square\square)\sqcup B(\phi)\big)\otimes B(\square) \\ &= \big(B(\square\square)\otimes B(\square)\big)\sqcup\big(B(\phi)\otimes B(\square)\big) \end{aligned}$$





and



$B(\phi)\otimes B(\square) \cong B(\square)$

$B(\square\square)\otimes B(\square) \cong B(\square\square\square)\sqcup B(\square)$

where $B(\square\square\square) = \{\rightarrow\rightarrow\rightarrow,\ \rightleftarrows\rightarrow,\ \rightleftarrows\leftarrow,\ \leftarrow\leftarrow\leftarrow\}$.

So far we have

$$\begin{tikzcd}[column sep=small]
B^{\otimes 0} & & B(\emptyset) \arrow[dr] & & & \\
B^{\otimes 1} & & & B(\square) \arrow[dl, dash] \arrow[dr, dash] & & \\
B^{\otimes 2} & & B(\emptyset) \arrow[dr, dash] & & B(\square\square) \arrow[dl, dash] \arrow[dr, dash] & \\
B^{\otimes 3} & & & B(\square) & & B(\square\square)
\end{tikzcd}$$

A <u>crystal</u> is a subset of $B^{\otimes \ell}$ closed under the action of $\tilde{e}$ and $\tilde{f}$.

The <u>crystal graph</u> of a crystal B is the graph with vertices B and edges $p - \tilde{f}p$.

A crystal is <u>irreducible</u> if its crystal graph is connected.

Let B be a crystal. The <u>character</u> of B is

$$ch(B) = \sum_{p \in B} x^{wt(p)}, \qquad \text{where}$$

$wt(p)$ is the endpoint of p.

A <u>highest weight path</u> is a path which is always ≥ 0.

If b is a highest weight path then $\tilde{e}p = 0$.

Our examples

$$B = B(\square) = \left\{ \begin{array}{c} \longrightarrow \\ \downarrow f \\ \longleftarrow \end{array} \right\} \quad \text{has} \quad \mathrm{char}(B(\square)) = x + x^{-1}$$

Then

$$B^{\otimes 2} = B \otimes B = \{ \rightarrow\rightarrow, \ \rightleftarrows, \ \leftrightarrows, \ \leftarrow\leftarrow \} \quad \text{has}$$

character $\mathrm{char}(B \otimes B) = (x + x^{-1})^2 = x^2 + 2 + x^{-2}$.

Next

$$B(\square\square) = \left\{ \begin{array}{c} \rightarrow\rightarrow \\ \downarrow f \\ \leftrightarrows \\ f \downarrow \\ \leftarrow\leftarrow \end{array} \right\} \quad \text{has character}$$

$$\mathrm{char}(B(\square\square)) = x^2 + 1 + x^{-2} = x^2 + x^0 + x^{-2}.$$

$B(\emptyset) = \{ \rightleftarrows \}$ has character $\mathrm{char}(B(\emptyset)) = x^0 = 1$.

and

$B^{\otimes 2} \cong B(\square\square) \sqcup B(\emptyset)$ and $\rightarrow\rightarrow$ and $\rightleftarrows$

are the highest weight paths in $B \otimes B$.

$$B(\square\square\square) = \left\{ \begin{array}{c} \rightarrow\rightarrow\rightarrow \\ f \downarrow \ \uparrow \tilde{e} \\ \leftrightarrows\!\rightarrow \\ f \downarrow \\ \rightleftarrows\!\leftarrow \\ f \downarrow \\ \leftarrow\leftarrow\leftarrow \end{array} \right\} \quad \text{has character}$$

$$\mathrm{char}(B(\square\square\square)) = x^3 + x + x^{-1} + x^{-3}.$$

and

$B^{\otimes 3} = B(\square\square\square) \sqcup B(\square) \sqcup B(\square)$ has highest weight

paths $\quad \rightarrow\rightarrow\rightarrow, \quad \rightarrow\rightleftarrows, \quad \rightleftarrows\!\rightleftarrows \quad$ and

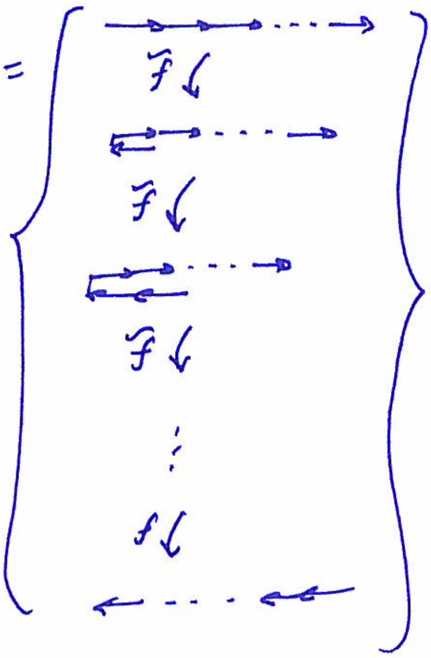
$$\mathrm{char}(B^{\otimes 3}) = (x + x^{-1})^3 = (x^3 + x + x^{-1} + x^{-3}) + (x + x^{-1}) + (x + x^{-1}).$$

Classification of irreducible $\mathfrak{sl}_2$-crystals

Theorem

(a) The irreducible $\mathfrak{sl}_2$-crystals are

$B(\underbrace{\square\square\square\square}_{k}) =$



with

$$\mathrm{char}(B(\overbrace{\square\square\square\square}^{k})) = x^k + x^{k-2} + \cdots + x^{-(k-2)} + x^{-k}$$

(b) Every crystal is a disjoint union of irreducible crystals.

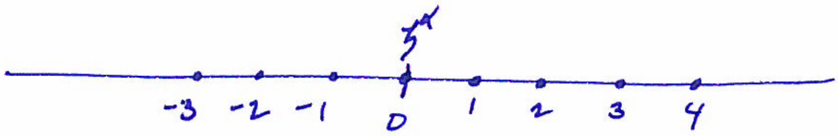
(c) Each irreducible crystal B has a unique highest weight path and

$$B \simeq B(\underbrace{\square\square\square\square\square}_{k}), \text{ if } p \text{ ends at } k.$$

$\mathfrak{sl}_3$-crystals

For $\mathfrak{sl}_3$-crystals the picture

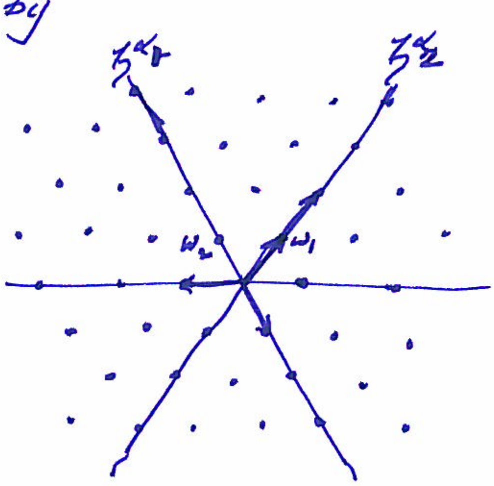
$\mathfrak{sl}_2$



with $\tilde{e}$ and $\tilde{f}$

is replaced by

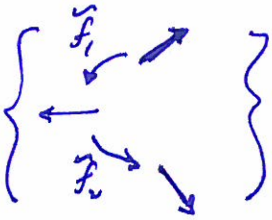
$\mathfrak{sl}_3$



with operators $\tilde{e}_1, \tilde{e}_2, \tilde{f}_1, \tilde{f}_2$.

Some examples:

$B(\square) =$

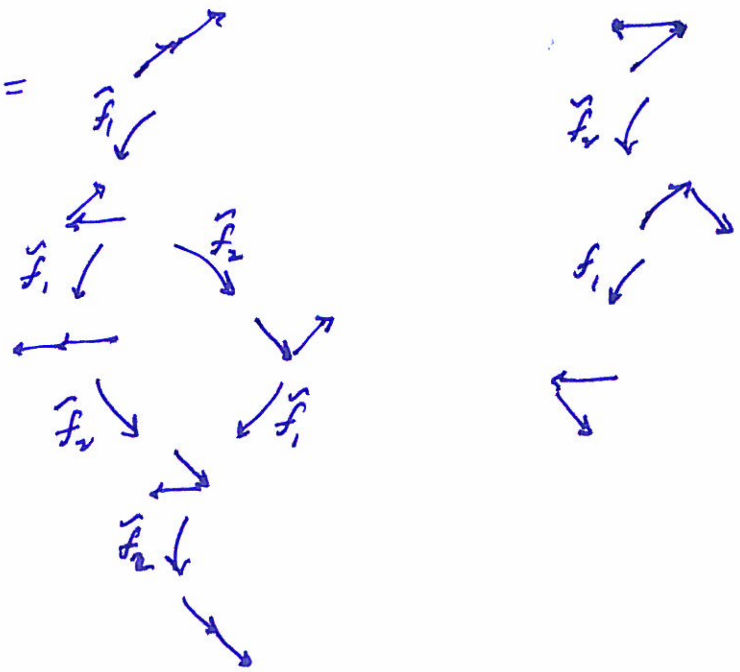


has character $\mathrm{char}(B(\square)) = x_1 + x_2 + x_3$.

Let $B = B(\square)$. Then

$$B^{\otimes 2} = B(\square) \otimes B(\square)$$

$=$



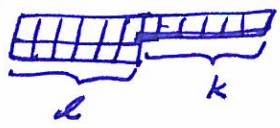
$= B(\square\square) \sqcup B\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$

The points of the positive/dominant chamber

$$P^+ = \{ k\omega_1^+ + l\omega_2 \mid k, l \in \mathbb{Z}_{\geq 0} \}$$

are in bijection with partitions with ≤ 2 rows.

$$P^+ \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{partitions with} \\ \leq 2 \text{ rows} \end{matrix} \right\}$$

$k\omega_1 + l\omega_2 \longmapsto$ 

We have

$$\mathrm{char}(B^{\otimes 2}) = (x_1 + x_2 + x_3)^2$$

$$= (x_1^2 + x_1 x_2 + x_3 x_1 + x_2^2 + x_3 x_2 + x_3^2)$$

$$+ (x_1 x_2 + x_1 x_3 + x_2 x_3), \quad \text{with}$$

$$\mathrm{char}(B(\square\square)) = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$= \sum_{1 \leq i \leq j \leq 3} x_i x_j, \quad \text{and}$$

$$\mathrm{char}(B(\begin{smallmatrix}\square\\\square\end{smallmatrix})) = x_1 x_2 + x_1 x_3 + x_2 x_3 = \sum_{1 \leq i < j \leq 3} x_i x_j .$$

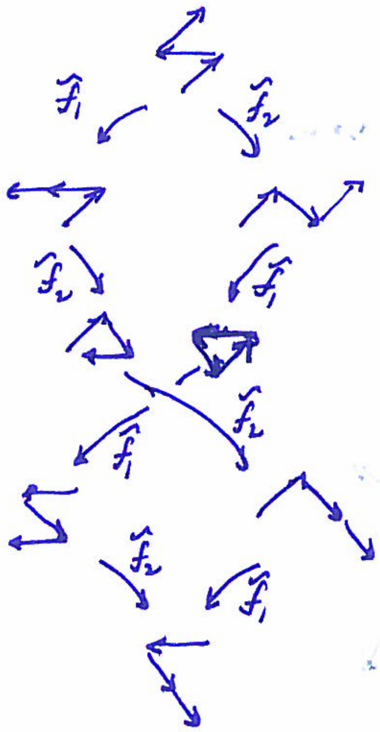
Now compute

$$B^{\otimes 3} = B(\square) \otimes B(\square) \otimes B(\square)$$

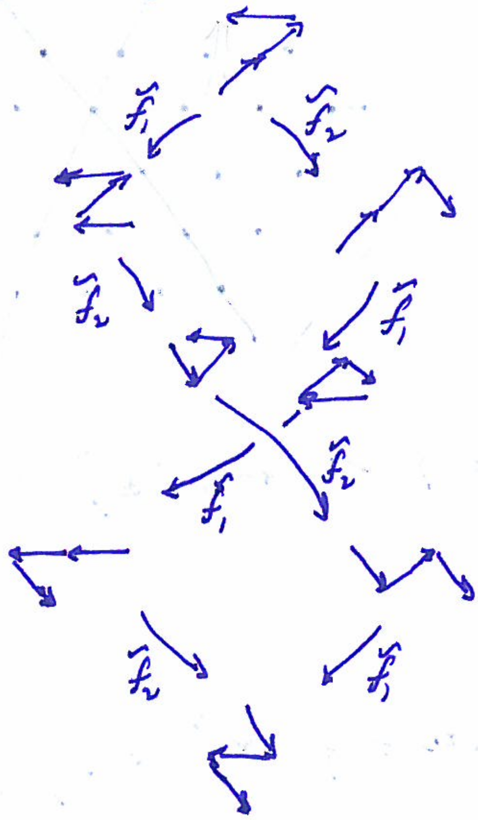
$$= B(\square\square) \otimes B(\square) \sqcup B(\begin{smallmatrix}\square\\\square\end{smallmatrix}) \otimes B(\square)$$

Three realizations of $B(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix})$

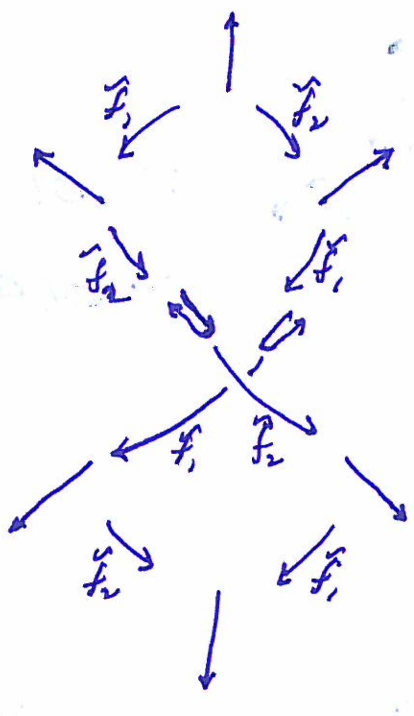
Inside $B(\begin{smallmatrix}\square\\\square\end{smallmatrix}) \otimes B(\square)$:



Inside $B(\square\square) \otimes B(\square)$:



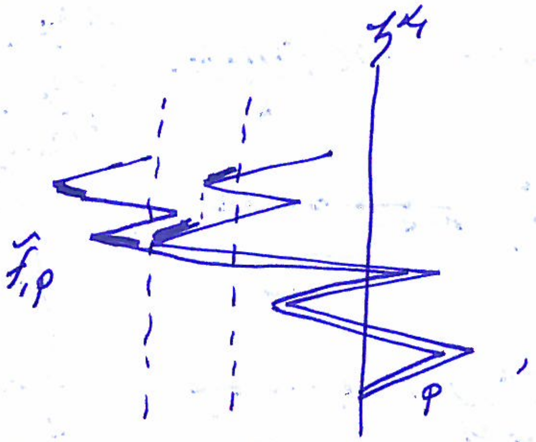
With the straight line path as highest weight path:



Definitions

A $\mathfrak{sl}_3$*-crystal* is a collection of paths closed under the root operators $\tilde{e}_1, \tilde{e}_2, \tilde{f}_1, \tilde{f}_2$.

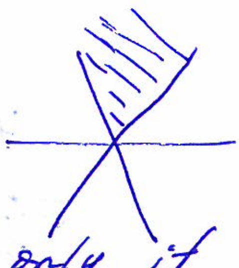
The root operators $\tilde{e}_1, \tilde{f}_1$ act like the $\mathfrak{sl}_2$-crystal operators $\tilde{e}, \tilde{f}$ in the $(\mathfrak{h}^{\alpha_1})^\perp$ projection



and $\tilde{e}_2, \tilde{f}_2$ act like the $\mathfrak{sl}_2$-crystal operators $\tilde{e}, \tilde{f}$ in the $(\mathfrak{h}^{\alpha_2})^\perp$ projection.

A *highest weight path* is a path p contained in

$$C = \cancel{\{x \in \}} \cap \begin{pmatrix}\text{positive}\\ \text{halfspaces}\end{pmatrix} =$$



A path p is highest weight if and only if $\tilde{e}_1 p = 0$ and $\tilde{e}_2 p = 0$.

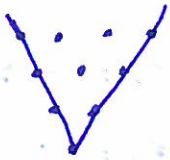
The *crystal graph* has edges labeled $\tilde{f}_1$ and $\tilde{f}_2$.

The crystal graph is *irreducible* if the crystal graph is connected.

Theorem

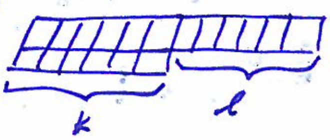
(a) The irreducible sl_3-crystals are indexed by the points in

$$P^+ = \{k\omega_1 + l\omega_2 \mid k, l \in \mathbb{Z}_{\geq 0}\}$$

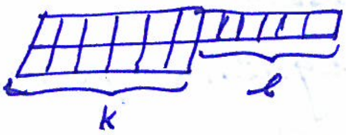


(b) Every sl_3 crystal is a disjoint union of irreducible crystals.

(c) Each irreducible crystal B has a unique highest weight path p and

$B \cong B($  $)$ if p ends at $k\omega_1 + l\omega_2$.

A column strict tableau of shape

$\lambda =$  is a filling of the boxes

of λ from $\{1, 2, 3\}$ such that

(a) the rows weakly increase (left to right)

(b) the columns strictly increase (top to bottom).

1	1	1	2	2	2	3
2	2	3	3			

Let $p_1 = \nearrow$, $p_2 = \leftarrow$, $p_3 = \searrow$. Let

$\lambda = k\omega_1 + l\omega_2$ and $p = \underbrace{p_1 p_1 \cdots p_1}_{k+l} \underbrace{p_2 p_2 p_2 \cdots p_2}_{k}$

There is a bijection from

$$\mathcal{B} = \begin{pmatrix} \text{the irreducible crystal with} \\ \text{highest weight path } p \end{pmatrix}$$

to

$$B(\lambda) = \left\{ \begin{matrix} \text{column strict tableaux of} \\ \text{shape } \lambda \text{ filled from } \{1,2,3\} \end{matrix} \right\}$$

given by reading the tableau in arabic reading order and taking the corresponding word in p_1, p_2, p_3

<u>Example</u>