Representation Theory Lecture 7, 9 September 2008.

Dual vector spaces

Let $\mathbb{F}$ be a field.

$\mathfrak{h}^* = \text{span}\{\omega_1, \ldots, \omega_n\}$ a vector space

$\mathfrak{h} = \text{Hom}(\mathfrak{h}^*, \mathbb{F})$ the dual vector space

Write
$$\langle \mu, \lambda^\vee \rangle = \mu(\lambda^\vee), \quad \text{for } \mu \in \mathfrak{h}^*, \ \lambda^\vee \in \mathfrak{h}.$$

Let $G = GL(\mathfrak{h}^*)$. G acts on $\mathfrak{h}^*$.
Define an action of G on $\mathfrak{h}$ by
$$\langle \mu, g\lambda^\vee \rangle = \langle g^{-1}\mu, \lambda^\vee \rangle.$$

Let $\omega_1, \ldots, \omega_n$ be a basis of $\mathfrak{h}^*$ and identify g with its matrix in $GL_n(\mathbb{F})$.
Let $\alpha_1^\vee, \ldots, \alpha_n^\vee$ be the dual basis in $\mathfrak{h}$. The matrix of the action of g on $\mathfrak{h}$ is
$$g^\vee = (g^t)^{-1}.$$

Reflections

A <u>reflection</u> is $s_\alpha \in GL(\mathfrak{h}^*)$ such that, in $GL_n(\mathbb{F})$,

s_α is conjugate to $\begin{pmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

with $\xi \in \mathbb{F}$, $\xi \neq 1$.

Then $s_\alpha^\vee \in GL(\mathfrak{h})$ is conjugate to $\begin{pmatrix} \xi^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

~~Let~~ Then

$$\mathfrak{h}^* = \mathfrak{h}^{\alpha^\vee} \oplus \mathbb{C}\alpha \quad \text{and} \quad \mathfrak{h} = \mathfrak{h}^\alpha \oplus \mathbb{C}\alpha^\vee$$

where

$$\mathfrak{h}^{\alpha^\vee} = (\mathfrak{h}^*)^{s_\alpha} = \{\mu \in \mathfrak{h}^* \mid s_\alpha \mu = \mu\} \quad \begin{pmatrix} 1\text{ eigenspace} \\ \text{of } s_\alpha \end{pmatrix}$$

$$\mathbb{C}\alpha = (\xi\text{-eigenspace of } s_\alpha)$$

$$\mathfrak{h}^\alpha = \mathfrak{h}^{s_\alpha^\vee} = \{\lambda^\vee \in \mathfrak{h} \mid s_\alpha^\vee \lambda^\vee = \lambda^\vee\} = \begin{pmatrix} 1\text{-eigenspace} \\ \text{of } s_\alpha \end{pmatrix}$$

$$\mathbb{C}\alpha^\vee = (\xi^{-1}\text{-eigenspace of } s_\alpha^\vee)$$

and

$$\mathfrak{h}^{\alpha^\vee} = \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha^\vee \rangle = 0\} \qquad \mathfrak{h}^\alpha = \{\lambda^\vee \in \mathfrak{h} \mid \langle \lambda^\vee, \alpha \rangle = 0\}.$$

Choose α and $\alpha^\vee$ so that $\langle \alpha, \alpha^\vee \rangle = 1 - \xi = 1 - \det(s_\alpha)$

Then

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_\alpha^{-1} \lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha \rangle \alpha^\vee.$$

<u>Check</u>: $s_\alpha \alpha = \alpha - \langle \alpha, \alpha^\vee \rangle \alpha = (1 - \langle \alpha, \alpha^\vee \rangle)\alpha = \xi \alpha,$

$$s_\alpha^{-1} \alpha^\vee = \alpha^\vee - \langle \alpha, \alpha^\vee \rangle \alpha^\vee = (1 - \langle \alpha, \alpha^\vee \rangle)\alpha^\vee = \xi \alpha^\vee,$$

as it should be.

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha = \mu - 0, \quad \text{if } \mu \in \mathfrak{h}^{\alpha^\vee}$$

$$s_\alpha^{-1} \lambda^\vee = \lambda^\vee - \langle \lambda^\vee, \alpha \rangle \alpha^\vee = \lambda^\vee - 0, \quad \text{if } \lambda^\vee \in \mathfrak{h}^\alpha.$$

Weyl groups

Let $\mathfrak{h}_{\mathbb{Z}}^*$ be a $\mathbb{Z}$-vector space.

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\text{-span}\{\omega_1, \ldots, \omega_n\},$$

where $\omega_1, \ldots, \omega_n$ is a $\mathbb{Z}$-basis of $\mathfrak{h}_{\mathbb{Z}}^*$.

$$\mathfrak{h}_{\mathbb{Q}}^* = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Q}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

$$\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^* = \mathbb{R}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

$$\mathfrak{h}_{\mathbb{C}}^* = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^* = \mathbb{C}\text{-span}\{\omega_1, \ldots, \omega_n\}$$

$$\mathfrak{h}_{\overline{\mathbb{Q}}}^* = \overline{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^* = \overline{\mathbb{Q}}\text{-span}\{\omega_1, \ldots, \omega_n\}.$$

A <u>Weyl group</u>, or <u>crystallographic</u> <u>reflection group</u>, is a finite subgroup W_0 of $GL(\mathfrak{h}_{\mathbb{Z}}^*)$ generated by reflections.

Let R^+ be an index set for the reflections in W_0 so that

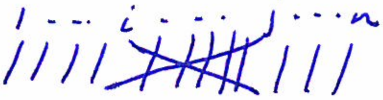
$$s_\alpha,\ \alpha \in R^+, \text{ are the reflections in } W_0$$

<u>WARNING</u>: A <u>Weyl group</u> is really a pair $(W_0, \mathfrak{h}_{\mathbb{Z}}^*)$. W_0 cannot exist without $\mathfrak{h}_{\mathbb{Z}}^*$.

<u>Examples</u> (Type GL_n)

$$\mathfrak{h}^*_{\mathbb{Z}} = \operatorname{span}\{\varepsilon_1, \ldots, \varepsilon_n\} \quad \text{with} \quad W_0 = S_n$$

acting by permuting $\varepsilon_1, \ldots, \varepsilon_n$. The reflections are

$$s_{ij} = s_{\varepsilon_i^\vee - \varepsilon_j^\vee} =$$  $$= \begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cccccc} & & i & & j & \end{array} \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & \cdots & 1 & \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array}$$

$$R^+ = \{(ij) \mid 1 \le i < j \le n\} \quad \text{or} \quad R^+ = \{\varepsilon_i^\vee - \varepsilon_j^\vee \mid 1 \le i < j \le n\}$$

and

$$\mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee} = (\mathfrak{h}^*)^{s_{ij}} = \{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid s_{ij}\mu = \mu\}$$

$$= \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \langle \mu, \varepsilon_i^\vee - \varepsilon_j^\vee \rangle = 0\}$$

$$= \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \mu_i = \mu_j\}.$$

The arrangement of hyperplanes

$$\mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee} \quad \text{in} \quad \mathfrak{h}^*_{\mathbb{C}}, \quad 1 \le i < j \le n$$

is the <u>braid</u> <u>arrangement</u>.

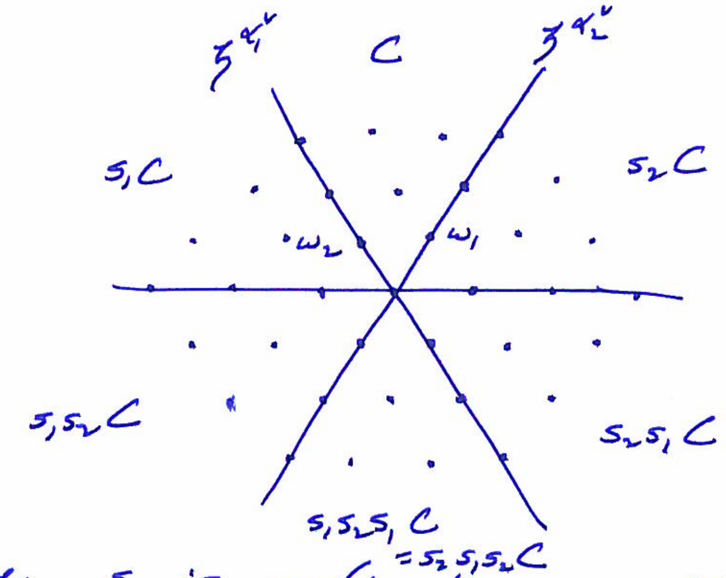
<u>Remark</u> $\operatorname{Conf}_n(\mathbb{C}^n) = \left(\mathfrak{h}^*_{\mathbb{C}} - \left(\bigcup_{1 \le i < j \le n} \mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee}\right)\right)$

has

$$\pi_1(\operatorname{Conf}_n(\mathbb{C}^n)) = \text{braid group}.$$

Example (Type SL_3)

$\mathfrak{h}^*_{\mathbb{Z}} = \mathrm{span}\{\omega_1, \omega_2\}$ and $W_0 = \left\langle s_1, s_2 \;\middle|\; \begin{array}{l} s_1^2 = s_2^2 = 1, \\ s_1 s_2 s_1 = s_2 s_1 s_2 \end{array} \right\rangle$



where s_1 is reflection in $\mathfrak{h}^{\alpha_1^\vee}$ and

s_2 is reflection in $\mathfrak{h}^{\alpha_2^\vee}$.

Let C be a fundamental chamber for the action of W_0 on $\mathfrak{h}^*_{\mathbb{R}}$.

$$W_0 \overset{1-1}{\longleftrightarrow} \{\text{chambers in } \mathfrak{h}^*_{\mathbb{R}}\}.$$

Let $\bar{C}$ be the closure of C.

The <u>dominant</u> <u>integral</u> <u>weights</u> are

$$P^+ = \mathfrak{h}^*_{\mathbb{Z}} \cap \bar{C} \quad \text{and} \quad P^{++} = \mathfrak{h}^*_{\mathbb{Z}} \cap C$$

are the <u>strictly</u> <u>dominant</u> <u>integral</u> <u>weights</u>.

There is a bijection

$$\begin{array}{ccc} P^+ & \longrightarrow & P^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{array}$$

where ρ is the point of P^{++} closest to 0.

Symmetric functions

Let

$$\mathcal{X} = \{X^\mu \mid \mu \in \mathfrak{h}_\mathbb{Z}^*\} \quad \text{with } X^\mu X^\nu = X^{\mu+\nu}.$$

$\mathcal{X}$ is the same group as $\mathfrak{h}_\mathbb{Z}^*$, except written multiplicatively. W_0 acts on $\mathcal{X}$ by

$$wX^\mu = X^{w\mu}, \quad \text{for } w \in W_0,\ \mu \in \mathfrak{h}_\mathbb{Z}^*.$$

Two one-dimensional representations of W_0 are

$$\begin{array}{ccc} W_0 \longrightarrow \mathbb{C} & \quad\text{and}\quad & W_0 \longrightarrow \mathbb{C} \\ w \longmapsto 1 & & w \longmapsto \det(w). \end{array}$$

The <u>ring of symmetric functions</u> is

$$\mathbb{C}[\mathcal{X}]^{W_0} = \{f \in \mathbb{C}[\mathcal{X}] \mid wf = f \text{ for all } w \in W_0\}$$

The vector space of <u>determinant symmetric functions</u> is

$$\mathbb{C}[\mathcal{X}]^{\det} = \{f \in \mathbb{C}[\mathcal{X}] \mid wf = \det(w) f, \text{ for all } w \in W_0\}.$$

<u>Example</u> (Type GL_3).

$$\mathfrak{h}_\mathbb{Z}^* = \text{span}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \quad \text{and} \quad \mathcal{X} = \{X^\mu \mid \mu \in \mathfrak{h}_\mathbb{Z}^*\}$$

where, for $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$,

$$X^\mu = X^{\mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3} = (X^{\varepsilon_1})^{\mu_1}(X^{\varepsilon_2})^{\mu_2}(X^{\varepsilon_3})^{\mu_3}$$

$$= x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3}, \quad \text{where } x_i = X^{\varepsilon_i}.$$

Then

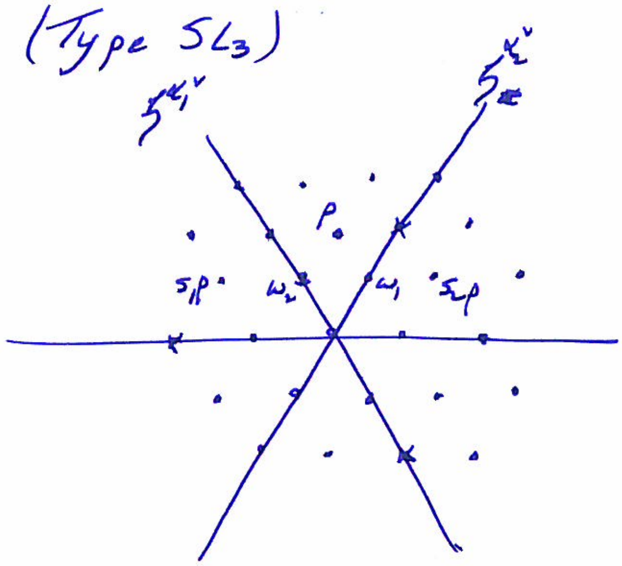
$$m_{(1,0,-2)} = x_1 x_2 x_3^{-2} + x_1 x_2^{-2} x_3 + x_1^{-2} x_2 x_3 \quad \text{is symmetric}$$

$$a_{(1,1,-2)} = x_1 x_2 x_3^{-2} - x_2 x_1 x_3^{-2} + x_1^{-2} x_2 x_3 - x_1 x_2^{-2} x_3 + x_2 x_3 x_1^{-2} + x_3 x_1 x_2^{-2}$$

is determinant symmetric.

since $\det(s_1) = -1$ and $\det(s_2) = -1$.

<u>Example</u> (Type SL_3)



$$m_\rho = X^\rho + X^{s_1\rho} + X^{s_2\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} + X^{s_1s_2s_1\rho} \quad \text{and}$$

$$m_{2\omega_1} = X^{2\omega_1} + X^{2\omega_2 - 2\omega_1} + X^{-2\omega_2} \quad \text{are symmetric and}$$

$$a_\rho = X^\rho - X^{s_1\rho} - X^{s_2\rho} + X^{s_1s_2\rho} + X^{s_2s_1\rho} - X^{s_1s_2s_1\rho} \quad \text{and}$$

$$a_{2\omega_1} = X^{2\omega_1} - X^{s_1 2\omega_1} - X^{s_2 2\omega_1} + X^{s_1s_2 2\omega_1} + X^{s_2s_1 2\omega_1} - X^{s_1s_2s_1 2\omega_1}$$

$$= X^{2\omega_1} - X^{2\omega_1} - X^{2\omega_2 - 2\omega_1} + X^{-2\omega_2} + X^{2\omega_2 - 2\omega_1} - X^{-2\omega_2}$$

are determinant symmetric.

Let

$$m_\mu = \sum_{\gamma \in W_0 \mu} X^\gamma, \qquad \text{for } \mu \in P.$$

Then $m_{w\mu} = m_\mu$, for $w \in W_0$ and the <u>orbit</u> <u>sums</u>, or <u>monomial</u> <u>symmetric</u> <u>functions</u>,

$$m_\lambda = \sum_{\gamma \in W_0 \lambda} X^\gamma, \qquad \lambda \in P^+$$

form a basis of $\mathbb{C}[X]^{W_0}$.

Let

$$a_\mu = \sum_{w \in W_0} \det(w^{-1}) X^{w\mu}, \qquad \text{for } \mu \in P.$$

Then

$$a_{v\mu} = \sum_{w \in W_0} \det(w^{-1}) X^{wv\mu}$$

$$= \sum_{w \in W_0} \det(v) \det((wv)^{-1}) X^{wv\mu}$$

$$= \det(v) a_\mu, \qquad \text{for } \mu \in P,\ v \in W_0.$$

If $\mu \in \mathfrak{Z}^{\alpha^\vee}$ so that $s_\alpha \mu = \mu$ then

$$a_\mu = a_{s_\alpha \mu} = \det(s_\alpha) a_\mu \text{ implies } a_\mu = 0,$$

since $\det(s_\alpha) \neq 1$. Thus

$$a_\mu = \sum_{w \in W_0} \det(w^{-1}) X^{\mu}, \qquad \mu \in P^{++},$$

form a basis of $\mathbb{C}[X]^{\det}$.

Boson-Fermion correspondence

$$\begin{array}{ccc} \mathbb{C}[X]^{W_0} & \longrightarrow & \mathbb{C}[X]^{\det} \\ f & \longmapsto & a_\rho f \end{array} \quad \text{is well defined}$$

since, if $f \in \mathbb{C}[X]^{W_0}$ then

$$w(a_\rho f) = (wa_\rho)(wf) = \det(w) a_\rho f, \quad \text{for } w \in W_0.$$

In fact, this map is invertible!

Let $g \in \mathbb{C}[X]^{\det}$,

$$g = \sum_{\mu \in P} g_\mu X^\mu, \quad \text{with } g_\mu \in \mathbb{C}.$$

Let s_α be a reflection in W_0. Then

$$\tfrac{1}{2}(g - s_\alpha g) = \tfrac{1}{2}(g - \det(s_\alpha) g) = \tfrac{1}{2}(g+g) = g \quad \text{and}$$

$$\begin{aligned} X^\mu - X^{s_\alpha \mu} &= X^\mu - X^{\mu - \langle \mu, \alpha^\vee \rangle \alpha} \\ &= X^\mu (1 - X^{-\langle \mu, \alpha^\vee \rangle \alpha}) \\ &= X^\mu (1 - X^{-\alpha})(1 + X^{-\alpha} + X^{-2\alpha} + \cdots + X^{-(\langle \mu, \alpha^\vee \rangle - 1)\alpha}). \end{aligned}$$

Hence

$$\frac{X^\mu - X^{s_\alpha \mu}}{1 - X^{-\alpha}} = X^\mu (1 + X^{-\alpha} + X^{-2\alpha} + \cdots + X^{-(\langle \mu, \alpha^\vee \rangle - 1)\alpha})$$

and

$$g = \tfrac{1}{2}(g - s_\alpha g) \text{ is divisible by } 1 - X^{-\alpha}.$$

It follows that

if $g \in \mathbb{C}[X]^{\det}$ then g is divisible by $\prod_{\alpha \in R^+} (1 - X^{-\alpha})$.

Claim:

$$a_\rho = X^\rho + \text{lower stuff}$$
$$= X^\rho \prod_{\alpha\in R^+} (1 - X^{-\alpha}).$$

Example: (Type GL_n)

$x_i = X^{\varepsilon_i}$, where $\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $W_0 = S_n$.

Then

$$a_\mu = \sum_{w\in S_n} \det(w^{-1}) X^{w\mu} = \sum_{w\in S_n} \det(w^{-1}) x_{w(1)}^{\mu_1} \cdots x_{w(n)}^{\mu_n}$$
$$= \det(x_i^{\mu_j})$$

In this case

$$C = \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \mu_i > \mu_j \text{ for } 1 \le i < j \le n\}$$

since $\mathfrak{h}^{\varepsilon_i^\vee - \varepsilon_j^\vee} = \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \mu_i = \mu_j\}$.

Hence

$$P^+ = \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \mu_i \in \mathbb{Z},\ \mu_1 \ge \mu_2 \ge \cdots \ge \mu_n\}$$
$$P^{++} = \{\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \mid \mu_i \in \mathbb{Z},\ \mu_1 > \mu_2 > \cdots > \mu_n\}$$

and

$$P^+ \longrightarrow P^{++}$$
$$\mu \longmapsto \mu + \rho \quad \text{where } \rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}$$

Hence

$$a_\rho = \det(x_i^{n-j}) = \begin{pmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ & & \vdots & & \\ x_n^{n-1} & & & & \end{pmatrix}$$