Representation Theory
7 October, 2008

0.0.1 Lie Groups

Definition 0.1. A **Lie group** is a group G that is also a **manifold** such that

$$\begin{array}{rcl} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_1 g_2 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of manifolds.

Definition 0.2. A **algebraic group** is a group G that is also a **variety** such that

$$\begin{array}{rcl} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_1 g_2 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of varieties.

Definition 0.3. A **topological group** is a group G that is also a **topological space** such that

$$\begin{array}{rcl} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_1 g_2 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of topological spaces.

Definition 0.4. A **group scheme** is a group G that is also a **scheme** such that

$$\begin{array}{rcl} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_1 g_2 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of schemes.

Definition 0.5. A **complex Lie group** is a group G that is also a **complex manifold** such that

$$\begin{array}{rcl} G \times G & \to & G \\ (g_1, g_2) & \mapsto & g_1 g_2 \end{array} \qquad \text{and} \qquad \begin{array}{rcl} G & \to & G \\ g & \mapsto & g^{-1} \end{array}$$

are morphisms of complex manifolds.

Note. Under these definitions, we can see that these objects are related. It should be noted that complex Lie group is not the same as Lie group. Hence, we can study the differences between $\mathbb{C}$ and $\mathbb{R}$ that are reflected in Lie group and complex Lie group.

Now,

Definition 0.6. A **manifold** is ... that is locally homeomorphic to open subset of $\mathbb{R}^n$.
A **complex manifold** is ... that is locally isoomorphic to open subset of $\mathbb{C}^n$.
A **variety** is ... that is locally isomorphic to closed set of algebraically closed field $\overline{\mathbb{F}}^n$.
A **scheme** is ... that is locally isomorphic to closed set of a ring R^n.

The main objective is to study an object like

$$GL_n(\ldots) := \{g \in M_{n\times n}(\ldots) \mid g \text{ is invertible}\},$$

where we can place any field or ring in .... We will state the main result first and slowly establish the links between two sets.

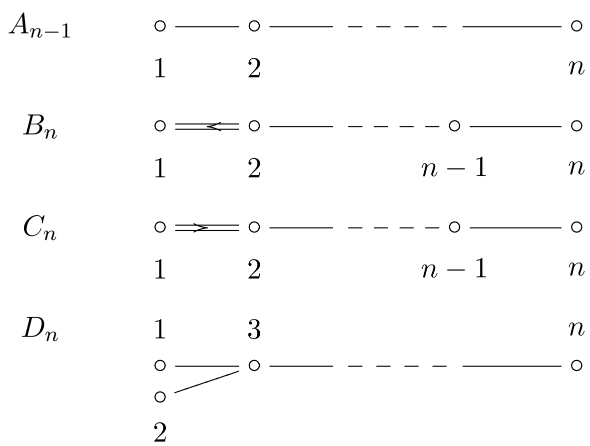
0.0.2 The Main Result

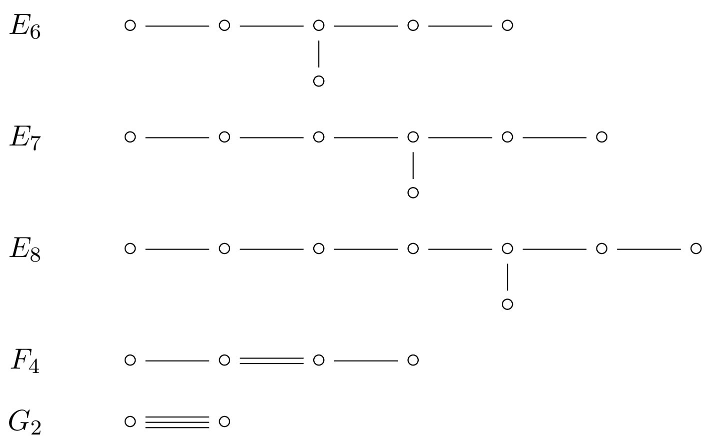
$$
\begin{array}{cc}
\left\{ \begin{array}{c} \text{connected reductive} \\ \text{complex algebraic groups} \end{array} \right\} & \\
\updownarrow & 1:1 \\
\left\{ \begin{array}{c} \text{connected compact} \\ \text{Lie groups} \end{array} \right\} & \\
\updownarrow & \text{almost } 1:1 \\
\left\{ \begin{array}{c} \text{complex semisimple} \\ \text{Lie algebra} \end{array} \right\} & \\
\updownarrow & \text{almost } 1:1 \\
\left\{ \begin{array}{c} \mathbb{Z}\text{-reflective groups} \\ (W_0, \mathfrak{h}^*_{\mathbb{Z}}) \end{array} \right\} & \\
\updownarrow & 1:1 \\
\left\{ \text{Dynkin diagrams} \right\} &
\end{array}
$$

We will start with Dynkin diagram.

0.0.3 Dynkin Diagram

A **Dynkin diagram** is one of the following pictures:





So, Dynkin diagram tells us almost everything about connected reductive complex algebraic groups.

0.0.4 $GL_n, SL_n, \mathcal{P}SL_n$

Let V be a vector space over field $\mathbb{F}$. Then

$$\begin{aligned} GL_n(\mathcal{F}) &:= \{g \in M_n(\mathbb{F}) \mid g \text{ is invertible}\}; \\ GL_n(V) &:= \{g \in End(V) \mid g \text{ is invertible}\}; \\ SL_n(\mathbb{F}) &:= \{g \in GL_n(\mathbb{F}) \mid \det g = 1\}. \end{aligned}$$

Notice that the group homomorphism $\det : GL_n(\mathbb{F}) \to \mathbb{F}^\times = GL_1(\mathbb{F})$ is a 1-dimensional representation of $GL_n(\mathbb{F})$. Further, $SL_n(\mathbb{F}) = \ker(\det)$, so $SL_n(\mathbb{F})$ is a normal subgroup of $GL_n(\mathbb{F})$. Hence, the sequence of group homomorphisms

$$1 \longrightarrow SL_n(\mathbb{F}) \longrightarrow GL_n(\mathbb{F}) \xrightarrow{\det} \mathbb{F}^\times \longrightarrow 1$$

is exact.

Definition 0.7. The **center** of GL_n is

$$Z(GL_n(\mathbb{F})) := \{c \cdot \mathrm{Id} \mid c \in \mathbb{F}^\times\} = \left\{ \begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix} \right\}$$

Define $\mathcal{P}GL_n$ to be

$$\mathcal{P}GL_n(\mathbb{F}) = \frac{GL_n(\mathbb{F})}{Z(GL_n(\mathbb{F}))}$$

Remark 1. This quotient group is often simple. If $\mathbb{F}$ is a finite field, then $\mathcal{P}GL_n$ classifies a family of finite simple groups.

Definition 0.8. The **center** of $SL_n(\mathbb{C})$ is

$$Z(SL_n(\mathbb{C})) := \{c \cdot \mathrm{Id} \mid c^n = 1\} = \mu_n = \{\text{nth root of } 1\}$$

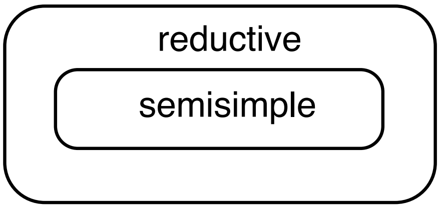
So, with these definitions in mind, we can think of the following objects in the following way:

$$\begin{aligned} \mathbb{F}^\times &= GL_1(\mathbb{F}) \\ \mu_n &= Z(SL_n(\mathbb{C})) \end{aligned}$$

Note. $SL_n(\mathbb{C}) \subseteq GL_n(\mathbb{C})$ and we can think of $GL_n(\mathbb{C}) = \mathbb{C}^\times \cdot SL_n(\mathbb{C})$. Now even though the sequence

$$1 \longrightarrow SL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times \longrightarrow 1$$

is exact, in general $SL_n(\mathbb{C}) \neq \mathcal{P}GL_n(\mathbb{C})$. This is because $\mathcal{P}GL_n(\mathbb{C})$ often don't have center. On the other hand, we have already shown that SL_n do.



Summary

GL_n is reductive, and not semisimple. SL_n and $\mathcal{P}GL_n$ are both semisimple. Dynkin diagrams for these are of the type A_{n-1}.

0.0.5 Unitrary, symplectic, and orthogonal groups

The **unitrary group** is

$$U_n := \{g \in GL_n(\mathbb{C}) \mid g\bar{g}^t = 1\},$$

where $\bar{g} = (\overline{g_{ij}})$, if $g = g_{ij}$ (i.e. conjugate matrix).

The **orthogonal group** is

$$\begin{aligned} O_n &:= \{g \in GL_n(\mathbb{C}) \mid gg^t = 1\} \\ SO_n &:= \{g \in SL_n(\mathbb{C}) \mid gg^t = 1\} \end{aligned}$$

The **simplectic group** is

$$Sp_{2n} := \{g \in GL_{2n}(\mathbb{C}) \mid gJg^t = J\},$$

where

$$J = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix} & 0 \end{array}\right) \text{ or } J = \left(\begin{array}{c|c} 0 & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & 0 \end{matrix} \\ \hline \begin{matrix} 0 & & -1 \\ & \iddots & \\ -1 & & 0 \end{matrix} & 0 \end{array}\right).$$

Note that different J's give different groups but they are isomorphic. Observe also that we can redefine O_n by $O_n := \{g \in GL_n(\mathbb{C}) \mid gJg^t = J\}$, where $J = \mathrm{Id}$. So, there must be generalisation of these groups. We will now explore alternative definitions.

Definition 0.9. Let V be a vector space over field $\mathbb{F}$. A **symmetric bilinear form** is a map

$$\begin{array}{rcl} \langle\ ,\ \rangle : V \times V & \to & \mathbb{F} \\ (v_1, v_2) & \mapsto & \langle v_1, v_2 \rangle \end{array}$$

such that

1. $\langle\ ,\ \rangle$ is bilinear over $\mathbb{F}$;
2. $\langle\ ,\ \rangle$ is symmetric (i.e. $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$.

So, we can define **orthogonal group** to be

$$\begin{array}{rcl} O_n & = & O(V) = O(V, \langle\ ,\ \rangle) = O(\langle\ ,\ \rangle) \\ & = & \{g \in GL(V) \mid \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle,\ \forall\, v_1, v_2 \in V\}. \end{array}$$

In other word, it is a group of linear transformation which preserves the metric.

Definition 0.10. A **skew symmetric bilinear form** is a map : $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ such that

1. $\langle\ ,\ \rangle$ is bilinear over $\mathbb{F}$;
2. $\langle v_1, v_2 \rangle = -\langle v_2, v_1 \rangle$.

Similarly, **simplectic group** is

$$\begin{array}{rcl} Sp_{2n} & = & Sp(V) = Sp(V, \langle\ ,\ \rangle) = Sp(\langle\ ,\ \rangle) \\ & = & \{g \in GL(V) \mid \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle,\ \forall\, v_1, v_2 \in V\}. \end{array}$$

Again, it is a group of linear transformation that preserves this metric. Now, let $\mathbb{F}$ be a field with an involution

$$\bar{\ } : \mathbb{F} \to \mathbb{F} \text{ sending } z \mapsto \bar{z}$$

Let V be a vector space over $\mathbb{F}$.

Definition 0.11. A **sesquilinear form** or **Hermition** on V is a map $\langle\ ,\ \rangle : V \times V \to \mathbb{F}$ such that

1. $\langle\ ,\ \rangle$ is **NOT** bilinear over $\mathbb{F}$, but instead
   - $\langle c_1 v_1 + c_2 v_2, v_3 \rangle = c_1 \langle v_1, v_3 \rangle + c_2 \langle v_2, v_3 \rangle$; and
   - $\langle v_1, c_2 v_2 + c_3 v_3 \rangle = \overline{c_2} \langle v_1, v_2 \rangle + \overline{c_3} \langle v_1, v_3 \rangle$.
2. $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$ for all $v_1, v_2 \in V$.

Then the **unitrary group** is

$$\begin{array}{rcl} U_n & = & U(V) = U(V, \langle\ ,\ \rangle) = U(\langle\ ,\ \rangle) \\ & = & \{g \in GL(V) \mid \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle,\ \forall\, v_1, v_2 \in V\}. \end{array}$$

0.0.6 Back to the main result

$$\left\{\begin{array}{c}\text{connected reductive}\\ \text{complex algebraic groups}\end{array}\right\} \xrightarrow[\text{1-1}]{\text{almost}} \left\{\begin{array}{c}\text{connected compact}\\ \text{Lie groups}\end{array}\right\}$$

$$G \longmapsto K$$

where K is a maximal compact subgroup. Note that maximal subgroup is enough to uniquely define G. This is due to the fact that K is almost dense in Zarisky Topology.

Example 1. $GL_n(\mathbb{C}) \mapsto U_n(\mathbb{C})$. For example,

$$\begin{aligned}\mathbb{C}^\times = GL_1(\mathbb{C}) \quad \mapsto \quad & U_1(\mathbb{C}) = \{z \in \mathbb{C} \mid z\bar{z} = 1\} = \{z \in \mathbb{C} \mid |z|^2 = 1\} \\ & = \{e^{2\pi\theta i} \mid 0 \leq \theta < 1\} = S^1\end{aligned}$$

Definition 0.12. A **torus** in an algebraic group G is a subgroup that is isomorphic to $\mathbb{C}^\times \times \ldots \times \mathbb{C}^\times$.
A **torus** in a compact Lie group K is a subgroup that is isomorphic to $S^1 \times \ldots \times S^1$ or $U_1 \times \ldots \times U_1$.

As an aside, torus in algebraic group is not compact, but torus in compact Lie group is. Now, with the definitions of tori in either group, we can look at the map between maximal tori. It turns out that the following diagram commutes:

$$\begin{array}{ccc} G & \xmapsto{\text{max compact}} & K \\ \downarrow & & \downarrow \\ T & \longmapsto & T_k \end{array} \qquad \downarrow \text{max torus}$$

Example 2.

$$\begin{array}{ccc} GL_n(\mathbb{C}) & \longmapsto & K = U_n(\mathbb{C}) \\ \downarrow & & \downarrow \\ \left\{ \left(\begin{array}{ccc} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{array} \right) \middle| x_i \in \mathbb{C}^\times \right\} & \longmapsto & \left\{ \left(\begin{array}{ccc} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{array} \right) \right\} \end{array}$$

The diagram commutes.

0.0.7 Weyl Group

Let G be a connected reductive algebraic group. Let T be a maximal torus in G. Let N be the normaliser of T. Then the Weyl group of G is $w_0 = N/T$.

Example 3. $G = GL_n$, $T = \left\{ \left(\begin{array}{ccc} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{array} \right) \right\}$ and
$N = \{n \times n\text{-matrix with one non-zero entry in each row and in each column }\}$.

Then we have

$$N \longmapsto N/T = w_0$$

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & \pi \\ e^3 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So, the Weyl group of GL_n is isomorphic to the group of permutations (i.e symmetric group).