Equivalences:

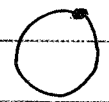
$$\left\{\begin{matrix}\text{connected, complex reductive} \\ \text{algebraic groups}\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{connected compact} \\ \text{Lie groups}\end{matrix}\right\}$$

$$GL_n(\mathbb{C}) \longmapsto U_n = \{ g \in GL_n \mid g\bar{g}^t = 1 \}$$

Examples

$$SL_n(\mathbb{C}) \longmapsto SU_2$$

$$\mathbb{C}^\times = GL_1(\mathbb{C}) \longmapsto U_1 = S^1 = \{ z \in \mathbb{C} \mid z\bar{z} = 1 \}.$$



not complex Lie group

= real Lie group.

i.e. locally $\cong$ to $\mathbb{R}^1$

$$\{1\} = SL_1(\mathbb{C}) \longmapsto SU_1 = \{1\}$$

Today

$$\{\text{Lie group}\} \xrightarrow[\text{not equiv.}]{\text{functor.}} \{\text{Lie algebras}\}$$

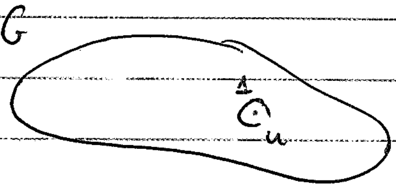
$$\left\{\begin{matrix}\text{connected} \\ \text{Lie group}\end{matrix}\right\} \xleftrightarrow{1:1} \{\text{Lie algebras}\}$$

but if res to connected we have 1-1.

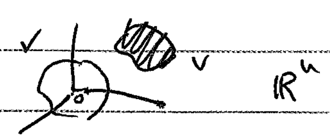
$$\left\{\begin{matrix}\text{connected compact} \\ \text{Lie group}\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{semisimple} \\ \text{Lie algebras}\end{matrix}\right\}.$$

Def^{n} A Lie group is group G that is also a manifold, i.e. G is locally isomorphic to $\mathbb{R}^n$

That is, $\exists$ $\varphi : \underset{\substack{\cap \\ G.}}{\overset{\text{open.}}{U}} \xrightarrow{\sim} \underset{\substack{\cap \\ \mathbb{R}^n}}{\overset{\text{open}}{V}}$ homeomorphisms. and group homomorphism



$$\xrightarrow{\varphi}$$



U open in G containing 1

V - open in $\mathbb{R}^n$. that contains 0.

i.e. map identity $\to 0$. So, nhd of $1 \xrightarrow{\varphi}$ nhd of 0.

exponential map

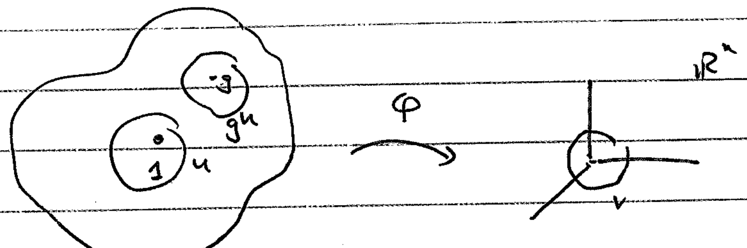
To define a Lie group we must have a homeomorphism

$$\exp : \mathfrak{g} \longrightarrow G \qquad \text{which is a } \overset{\text{meo}}{\text{homorphism}}$$

$$0 \longmapsto 1 \qquad \text{and homeom in the nhd of } 0.$$

where $\mathfrak{g}$ is an $\mathbb{R}$-vector space.

If U is an open nhd of 1 in G then gU is an open nhd of $g \in G$.



So, we only need to define exp for a nhd around 1.

$$V \xleftarrow{\sim} U \xrightarrow{\sim} gU$$
g invertible

Lie algebras

Let G be a Lie group. The ring of functions of G is

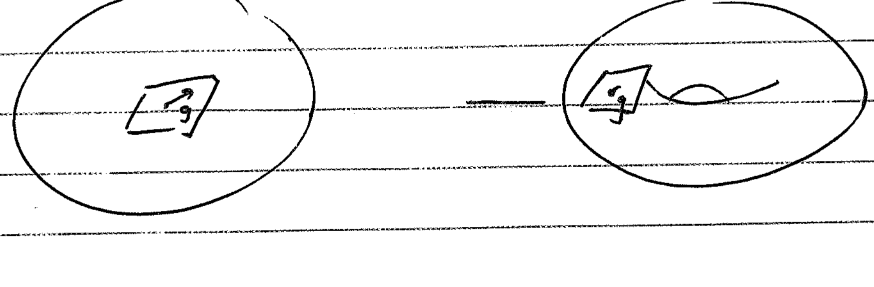
$$C^\infty(G) = \{ f : G \to \mathbb{R} \mid f \text{ is smooth at } g \ \forall g \in G \}$$

i.e. $\left(\frac{d^k f}{dx^k}\right)\Big|_{x=g}$ exists $\forall\, k \in \mathbb{Z}_{>0}$

Let $g \in G$. A tangent vector to G at g is a linear map $\eta_g : C^\infty(G) \longrightarrow \mathbb{R}$ such that

$$\eta_g(f_1 f_2) = \eta_g(f_1)\, f_2(g) + f_1(g)\, \eta_g(f_2) \qquad (\text{No function})$$

for $f_1, f_2 \in C^\infty(G)$ (η_g is the derivative at the point g)

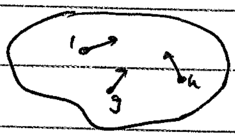


A vector field is a linear map $\eta : C^\infty(G) \to C^\infty(G)$ such that

- $\eta(f_1 f_2) = \eta(f_1) f_2 + f_1 \eta(f_2)$ for $f_1, f_2 \in C^\infty(G)$

It takes $f_u \to f_u$. i.e. η is a derivation of $C^\infty(G)$

$\eta_g(f) = (\eta f)(g)$



A left invariant vector field is a vector field $\eta : C^\infty(G) \to C^\infty(G)$ such that

$$L_g \circ \eta = \eta \circ L_g \qquad \text{for } g \in G$$

where $L_g : C^\infty(G) \to C^\infty(G)$ given by $(L_g f)(x) = f(g^{-1}x)$

~~g~~ This defines the action of group on a field of functions.

The Lie algebra of G is the vector space

$$\mathfrak{g} = \{ \text{left invariant } \cancel{\text{over}} \text{ vector fields over } G \}$$

with bracket

$$[\eta, \psi] = \eta \circ \psi - \psi \circ \eta$$

($\mathfrak{g} = T_1(G) = \{ \text{tangent vector to } G \text{ at } 1 \}$ – only need to define at tan vector at 1.

tangent vector to G at doesn't have natural bracket.
as $\eta_1 : C^\infty(G) \to \mathbb{R}$. – can't compose

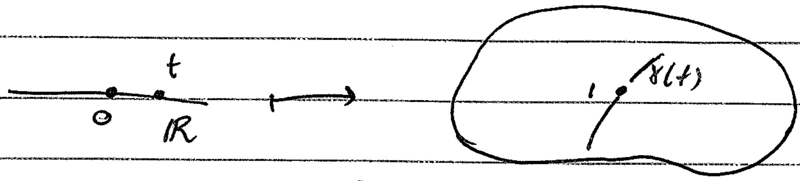
There is a vector space isomorphism.

$$\left\{ \begin{matrix} \text{left invariant} \\ \text{vector fields} \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{tangent vector} \\ \text{to } G \text{ at } 1 \end{matrix} \right\}$$

where $\eta_1(f) = (\eta f)(1)$ $\quad \eta \longmapsto \eta_1$

Where is exp map?

A one parameter subgroup of group G is a smooth homomorphism
$\gamma : \mathbb{R} \longrightarrow G$



This defines a curve in G.

Examples $G = GL_n$

① $x_{ij} : \mathbb{R} \longrightarrow GL_n(\mathbb{R})$ with $x_{ij}(t) = \begin{array}{cc} & j^{th} \\ i - & \begin{pmatrix} 1 & & t \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \end{array}$

$x_{12}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ $\quad x_{12}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ $\quad x_{12}(t)\, x_{12}(s) = \begin{pmatrix} 1 & t+s \\ 0 & 1 \end{pmatrix}$
$= x_{12}(s+t)$

So x_{ij} corresponds to elementary ~~do~~ column operations.
$\Leftrightarrow$ one ~~-~~ parameter subgroup.

② $h_i : \mathbb{R} \rightarrow GL_n(\mathbb{R})$ given by $\begin{array}{cc} & i^{th} \\ i^{th} & \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & e^t & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \end{array}$

$h_i(t)\, h_i(s) = h_i(t+s)$

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & e^{s+t} & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & e^{t+s} & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

GL_n is generated by elementary matrices ~~= GL_n is~~
$= GL_n$ " " " 1-parameter subgroups.
$= GL_n$ " " " a nhd of 1.

Let $\gamma : \mathbb{R} \longrightarrow G$ be a 1-parameter subgroup in G
Define ~~d~~ ~~on~~

$$\left.\frac{d\, f(\gamma(t))}{dt}\right|_t = \lim_{h \to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}.$$

You can take a derivative in $\mathbb{R}^n$. This defines a tangent vector.

We have a ~~vector~~ space isomorphism

$$\left\{ \begin{matrix} \text{one parameter} \\ \text{subgroups of } G \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{tangent vectors} \\ \text{to } G \text{ at } 1 \end{matrix} \right\}$$

$$\gamma \longmapsto \gamma_1$$

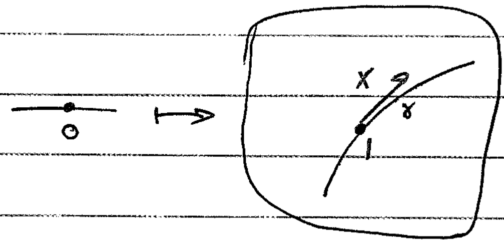
where

$$\gamma_1 f = \left.\frac{d\, f\circ\gamma}{dt}\right|_{t=0} = \lim_{h\to 0} \frac{f\circ\gamma(h) - f\circ\gamma(0)}{h}$$

Note $\gamma : \mathbb{R} \to G$ $\quad f : C^\infty(G) \to \mathbb{R}$.

The exponential map is

$$\exp : \mathfrak{g} \longrightarrow G$$
$$tX \longmapsto e^{tX}$$



where $e^{tX} = \gamma(t)$

So, $\exp(0) = \gamma(0) = 1$.

Examples The Lie algebra $\mathfrak{gl}_n$ is

$$\mathfrak{gl}_n = \{ x \in M_n(\mathbb{C}) \} \text{ with bracket.}$$

$$[x,y] = xy - yx$$

The exponential map is $\exp : \mathfrak{gl}_n \longrightarrow GL_n$

$$tX \longmapsto e^{tX}$$

where $e^{A} = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$ for $A \in M_n(\mathbb{C})$

$\mathfrak{gl}_n$ has basis $\{ E_{ij} \mid 1 \le i,j \le n \}$. Then

$$\exp(E_{ij}) = 1 + tE_{ij} + \frac{t^2 E_{ij}^2}{2!} + \ldots$$

$$= 1 + \begin{pmatrix} 0 & & t \\ & \ddots & \\ & & 0 \end{pmatrix} + \frac{t^2}{2!}\cdot 0 + \frac{t^3}{3!}\cdot 0 + \ldots \quad \text{if } i \neq j$$

$$= \begin{bmatrix} 1 & & t \\ & \ddots & \\ & & 1 \end{bmatrix} = x_{ij}(t)$$

if $i = j \Rightarrow$ $= 1 + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} t + \frac{t^2}{2!} \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \ldots$

$$= \begin{pmatrix} 1 & & \\ & e^t & \\ & & 1 \end{pmatrix} = h_i(e^t)$$

Rudin Real & Complex Analysis

The point: All connected Lie groups are generated by "elementary matrices" and these are the images of your favourite basis of the Lie algebra $\mathfrak{g}$.

Examples: $\exp : \mathfrak{g} \to G$ homeomorphisms in the nbd of 0

$\exp : gl_1 \to GL_1$ is $\exp . \mathbb{C} \to \mathbb{C}^\times$ so $\mathbb{C} \cong \mathbb{C}$ locally.

What is the $\widehat{\exp}$ map? We want it to be a smooth hom^m

$$e : \mathbb{C} \to \mathbb{C}^\times$$

i.e.

$$e(z) = a_0 + a_1 z + a_2 z^2 + \dots \quad \text{- smooth}$$

$$e(x+z) = \cancel{e(x)e(z)} = a_0 + a_1 x + a_1 z + a_2 x^2 + 2a_2 xz + a_2 z^2 + a_3 x^3 + 3a_3 x^2 z + 3a_3 x z^2 + a_3 z^3$$

GAGA Want $e(x+z) = e(x)e(z)$

wo

$$e(x)e(z) = a_0^2 + a_1 a_0 x + a_0 a_1 z + a_2 a_0 x^2 + a_1^2 xz + a_0 a_2 z^2 + a_3 a_0 x^3 + a_2 a_1 x^2 z + a_1 a_2 x z^2 + a_0 a_3 z^3$$

$a_1 a_0 = a_1 \qquad a_1^2 = 2a_2 \qquad a_1 a_2 = 3a_3 \qquad a_1 a_3 = 4a_4$

$\Rightarrow \quad a_0 = 1 \qquad a_2 = \frac{a_1^2}{2} \qquad a_3 = \frac{a_1^3}{3!} \qquad a_4 = \frac{a^4}{4!} \quad \dots$

So, in the world of formal power series the only homo^m $\mathbb{C} \to \mathbb{C}^\times$ are $e^{a_1 z}$. This is the local homo^m ~~appears in most~~..

Example. $SL_2(\mathbb{C}) = \{ g \in GL_2(\mathbb{C}) \mid \det g = 1 \}$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$$

has Lie algebra

$$sl_2 = \{ x \in M_2(\mathbb{C}) \mid tr\, x = 0 \}$$

with basis

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \cancel{z} h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $[x,y] = h \qquad [h,x] = 2x \qquad [h,y] = -2y$

The maximal compact subgroup of $SL_2(\mathbb{C})$ is.

$$SU_2 = \{ g \in SL_2(\mathbb{C}) \mid g\bar{g}^t = 1 \}$$

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C}, \; |a|^2 + |b|^2 = 1 \right\}$$

from det $= 1$

The Lie algebra

$$su_2 = \{ x \in M_2(\mathbb{C}) \mid tr\, x = 0 \quad x + \bar{x}^t = 0 \}$$

has basis $\{ i\sigma^x, i\sigma^y, i\sigma^z \}$ where ~~$\sigma^z =$~~

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the Pauli matrixes. If $A \in su_2$ then $e^A \in SU_2$.

su_2 is a $\mathbb{R}$-vector space (SU_2 is a Lie group locally $\cong$ to $\mathbb{R}^3$) NOT ~~compact~~plex

The complexification of su_2

$$\mathbb{C} \otimes_{\mathbb{R}} su_2 = \mathbb{C}\text{ span}\{ i\sigma^x, i\sigma^y, i\sigma^z \} = sl_2(\mathbb{C}) \neq SU_2$$

$\uparrow$ dim $= 1$ $\qquad\qquad$ $\uparrow$ dim 3.

The point

$$SL_2(\mathbb{C}) \longmapsto SU_2 \qquad \text{maximal compact.}$$

$$sl_2 = \mathbb{C} \otimes_{\mathbb{R}} su_2 \longleftarrow su_2$$