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Eg. Equivalences:

$$\left\{\begin{array}{l}\text{conn. complex}\\ \text{reductive algebraic}\\ \text{groups}\end{array}\right\} \qquad \left\{\begin{array}{l}\text{compact}\\ \text{Lie groups}\end{array}\right\}$$

$$GL_n(\mathbb{C}) \longmapsto U_n$$

$$\mathbb{C}^\times,\ \mathbb{C}^\times \times \mathbb{C}^\times \qquad S^1,$$ 

$$\left\{\begin{array}{l}\text{groups that}\\ \text{also "spaces"}\end{array}\right\} \longrightarrow \left\{\text{Lie Algebras}\right\}$$

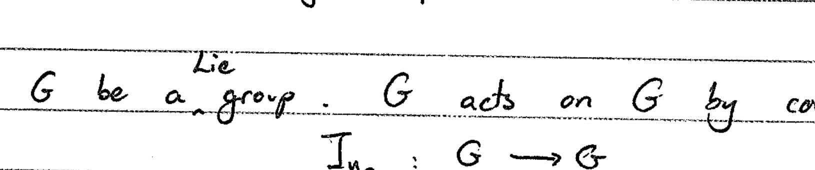
$$GL_n(\mathbb{C}) \longmapsto gl_n = M_n(\mathbb{C})$$

Today: $\left\{\begin{array}{l}\text{reductive}\\ \text{Lie algebras}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\mathbb{Z}\text{ reflection}\\ \text{group } (W_0, h^*_{\mathbb{Z}})\end{array}\right\}$

If $f: X \to Y$ is a morphism of spaces, this gives

$$df: T_x(X) \longrightarrow T_{f(x)}(Y) \qquad \text{for } x \in X$$

a map between tangent spaces.



So, when you have a map between spaces, we also get a map between tangent spaces.

Let G be a Lie group. G acts on G by conjugation

$$I_{n_g}: G \longrightarrow G \qquad \text{for } g \in G.$$
$$h \mapsto ghg^{-1}$$

So, G acts on $\mathfrak{g} = T_1(G)$ by the Adjoint action

$$Ad_g: \mathfrak{g} \to \mathfrak{g}$$
$$Ad_g = d(I_{n_g})$$

$\mathfrak{g} = Lie(G)$ Lie algebra of G.

Example $G = GL_n(\mathbb{C})$ has Lie algebra

$$\mathfrak{g} = gl_n(\mathbb{C}) = \{x \in M_n(\mathbb{C})\} \quad \text{with}$$

exponential map $\exp : \mathfrak{gl}_n \longrightarrow GL_n$

$$x \longmapsto e^x \qquad \text{where}$$

$$tx \longmapsto e^{tx}$$

$$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

~~How does e^{tx} acts on G (conjugation action).~~

How does G act on $\mathfrak{g}$ (Adjoint action)

$$I_{n_g}(e^{tx}) = g e^{tx} g^{-1} = g\left(1 + tx + \frac{tx^2}{2!} + \dots\right)g^{-1}$$
$$= e^{t(gxg^{-1})}$$

So, $Ad_g(x) = gxg^{-1}$; i.e. $Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g}$

$$x \longmapsto gxg^{-1}$$

Note that : SO_n, O_n, Sp_n, U_n, ... are subgroups of GL_n. So, we can work out the action of these groups.

G acts on G by conjugation.

G acts on $\mathfrak{g}$ by Adjoint action. (so $\mathfrak{g}$ is a G-module)

Let M be a G-module ; M is a vector space and G acts on M, Let

$$\rho : G \longrightarrow GL(M) = End(M)$$
$$g \longmapsto \rho(g)$$

be the corresponding representation. This gives.

$$d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}_n(M) = End(M)$$

a representation of Lie algebra $\mathfrak{g}$, Abusing notation ρ "=" $d\rho$.

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}_n(M) \qquad \rho : G \longrightarrow GL(M)$$
$$x \longmapsto \rho(x) \qquad \text{where} \qquad e^{tx} \mapsto \rho(e^{tx}) = e^{t\rho(x)}$$

where $\rho(x)$ is the element of the Lie algebra corresponding to the one ~~dimensional~~ parameter subgroup $\rho(e^{tx})$ (~~by~~ inside $GL(M)$)

So, M is a $\mathfrak{g}$-module.

So, $\mathfrak{g}$ acts on $\mathfrak{g}$ by adjoint action :

$$ad_g : \mathfrak{g} \to \mathfrak{g} \quad \text{for } g \in \mathfrak{g}$$

coming from $Ad_{e^{ty}} : \mathfrak{g} \to \mathfrak{g}$

$$x \mapsto e^{ty} x e^{-ty}$$

Claim : $ad_y : \mathfrak{g} \to \mathfrak{g}$

$$x \mapsto [y, x] = yx - xy$$

<u>Because</u> : $Ad_{e^{ty}}(x) = e^{ty} x e^{-ty} = \left(1 + ty + \frac{t^2y^2}{2!} + \dots\right) x \left(1 - ty + \frac{t^2y^2}{2!} + \dots\right)$

$$= x + t(yx - xy) + \frac{t^2}{2!}(y^2x - 2yxy + xy^2) + \dots$$

$$= Id(x) + t\, ad_y(x) + \frac{t^2}{2!}(ad_y)^2(x) + \dots$$

Since, $(ad_y)^2(x) = ad_y([y,x]) = [y,[y,x]] = [y, yx - xy]$

$$= y(yx - xy) - (yx - xy)y$$

$$= y^2x - 2yxy + xy^2$$

So, $Ad_{e^{ty}}(x) = e^{t\, ad_y}(x)$

<u>Summary</u>

G acts on G by <u>conjugation</u> $\quad Ing : G \to G : k \to gkg^{-1}$

G acts on $\mathfrak{g}$ by <u>Adjoint action</u> $\quad Ad_g : \mathfrak{g} \to \mathfrak{g} : x \to gxg^{-1}$

$\mathfrak{g}$ acts on $\mathfrak{g}$ by <u>adjoint action</u> $\quad ad_y : \mathfrak{g} \to \mathfrak{g} \quad$ for $g \in G$

$$: x \to [y, x]$$

Let M be a G-module. (M is also a $\mathfrak{g}$-module)

The dual to M is

$$M^* = Hom(M, \mathbb{C}) = \{\varphi : M \to \mathbb{C} \mid \varphi \text{ is linear}\}$$

with G action and $\mathfrak{g}$-action

$(g\varphi)(m) = \varphi(g^{-1}(m))$ for $g \in G$, $m \in M$.

$(x\varphi)(m) = \varphi(-xm)$ for $x \in \mathfrak{g}$, $m \in M$

since $(e^{xt})^{-1} = e^{-tx} = e^{t(-x)}$

So, we get

$\mathfrak{g}$ acts on $\mathfrak{g}^*$ by coadjoint action

G acts on $\mathfrak{g}^*$ by coAdjoint action

Tori and Cartan ~~Subgroup~~ Subalgebras.

Let G be an algebraic group. A torus is G is a subgroup H isomorphic ~~of~~ to $\mathbb{C}^\times \times \dots \times \mathbb{C}^\times = GL_1 \times \dots \times GL_1$.
Let K be a compact Lie group. A torus in K is a subgroup T isomorphic to $S^1 \times \dots \times S^1 = U_1 \times \dots \times U_1$.

Let $\mathfrak{g}$ be a Lie algebra. A abelian subalgebra is a subalgebra $\mathfrak{h}$ such that

$$[h_1, h_2] = 0 \quad \text{for } h_1, h_2 \in \mathfrak{h}$$

(if $\mathfrak{h} \subseteq \mathfrak{gl}_n$ then $0 = [h_1, h_2] = h_1 h_2 - h_2 h_1$ means h_1 and h_2 commute)

A Cartan subalgebra is a maximal abelian subalgebra

Examples: A maximal torus in GL_n is

$$H = \left\{ \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \;\middle|\; x_1, \dots, x_n \in GL_1 \right\}$$

if we want to get the other we can just conjugate it (Sylow's thm)

A Cartan subalgebra of $\mathfrak{gl}_n$ is

$$\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \;\middle|\; h_i \in \mathbb{C} \right\}$$

$\mathfrak{h} = Lie(H)$

$$e^{\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix}} = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}$$

The only way we know to get information about G is via H.

Irreducible representations of H

(All irreducible representations of a commutative algebra) are 1-dimensional (over $\mathbb{C}$)

(equiv to Jordan Normal Form)

The irreducible (rational) representations of H are

$$X^{\mu} = X^{\mu_1\varepsilon_1 + \dots \mu_n\varepsilon_n} = X^{\mu_1\varepsilon_1} \cdots X^{\mu_n\varepsilon_n} = (X^{\varepsilon_1})^{\mu_1} \cdots (X^{\varepsilon_n})^{\mu_n}$$

where $\mu_i \in \mathbb{Z}$

and $X^{\varepsilon_i} : H \longrightarrow \mathbb{C}^* = GL_1(\mathbb{C})$ $\quad\Big|\quad$ $X^{\mu} : H \longrightarrow \mathbb{C}^*$

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \longmapsto x_i \qquad \Bigg| \qquad \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \longmapsto x_1^{\mu_1} \cdots x_n^{\mu_n}$$

The irreducible representations of $\mathfrak{h}$ are

$$\mu = \mathfrak{h} \longrightarrow \mathbb{C} = \mathfrak{gl}_1$$

$$\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \longmapsto \mu_1 h_1 + \dots + \mu_n h_n$$

so that $\mu = \mu_1\varepsilon_1 + \dots + \mu_n\varepsilon_n$ with $\varepsilon_i : \mathfrak{h} \longrightarrow \mathbb{C}$

$$\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \longmapsto h_i$$

So, $\mathfrak{h}^* = \{\text{linear maps } \mu : \mathfrak{h} \to \mathbb{C}\}$ is the set of irred reps of $\mathfrak{h}$.

Toward Weyl groups

Let M be a G-module. $H \subseteq G$ so H acts on M. i.e M is an H-module (and an $\mathfrak{h}$-module).

Let $\mu \in \mathfrak{h}^*$ (an irreducible representation of $\mathfrak{h}$). The μ-weight space of M is:

$$M_{\mu} = \{ m \in M \mid \text{for each } t \in H,\ t \cdot m = X^{\mu}(t)\, m \}$$

$$= \{ m \in M \mid \text{for each } h \in \mathfrak{h},\ h \cdot m = \mu(h)\, m \}$$

i.e. ~~H, $\mathfrak{h}$ acts on~~ m are eigenvectors of H & $\mathfrak{h}$.

The

The generalised μ-weight space of M is

$$M_\mu^{gen} = \{ m \in M \mid \text{for each } t \in H,\ (t - \chi^M_\mu(t))^l m = 0 \text{ for some } l \in \mathbb{Z}_{\geq 0} \}$$
$$= \{ m \in M \mid \text{for each } (h - \mu(h))^l m = 0 \text{ for some } l \in \mathbb{Z}_{\geq 0} \}$$

Basically we add in Jordan normal form as well.

$$\left(\begin{pmatrix} \mu & 1 & 0 \\ & \ddots & 1 \\ 0 & & \mu \end{pmatrix} - \begin{pmatrix} \mu & & 0 \\ & \ddots & \\ 0 & & \mu \end{pmatrix} \right)^l m = 0$$

$M_\mu^{gen} \neq 0 \implies M_\mu \neq 0 \qquad M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu^{gen}$

(Jordan normal form)

A weight of M is μ such that that $M_\mu^{gen} \neq 0$

We will try to understand M_μ^{gen} by understanding its weight.

Question: What are the weights of the adjoint representions.

$\mathfrak{g}$ acts on $\mathfrak{g}$: $\mathfrak{g}_0 = \{ x \in \mathfrak{g} \mid [h, x] = 0(h)x \ \forall h \in \mathfrak{h} \}$
$= \{ x \in \mathfrak{g} \mid [h, x] = 0 \ \forall \ h \in \mathfrak{h} \}$
$= \mathfrak{h}.$

"$\supseteq$" h commutes with everything in $\mathfrak{h}$.

"$\subseteq$" $\mathfrak{h}$ is maximal.

The roots (or root system) of $\mathfrak{g}$ are the non-zero weights of $\mathfrak{g}$.

The Weyl group of G is

$$W_0 = N(H)/H \quad \text{where } N(H) = \{ n \in G \mid nHn^{-1} = H \} \supseteq H$$

N is the stabilizer of the maximal abelian subgroup H.

(the "interesting part" is $N(H)$)

W_0 acts on H by $nhn^{-1} = h^n$ for $n \in N(H)$ $h \in H$
$\Rightarrow$ W_0 acts on $\mathfrak{h}$
$\Rightarrow$ W_0 acts on $\mathfrak{h}^*$

If M is a G-module, then $N(H)$ acts on M and

$$w: M_\mu \longrightarrow M_{w\mu}, \qquad w \in W_0 \qquad (1)$$

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu^{gen} \qquad (2)$$

(1), & (2) are tools for study representations. So, if we understand (1) & (2) we know everything about M.